THE LAW OF GRAVITATION
IN RELATIVITY

THE UNIVERSITY OF CHICAGO PRESS
CHICAGO, ILLINOIS

THE BAKER & TAYLOR COMPANY
NEW YORK

THE CAMBRIDGE UNIVERSITY PRESS
LONDON

THE MARUZEN-KABUSHIKI-KAISHA
TOKYO, OSAKA, KYOTO, FUKUOKA, SENDAI

THE COMMERCIAL PRESS, LIMITED
SHANGHAI

THE LAW OF GRAVITATION IN RELATIVITY

By
HORACE C. LEVINSON, PH.D.
and
ERNEST BLOOMFIELD ZEISLER, PH.D.

Second Edition

M. N. Saha

THE UNIVERSITY OF CHICAGO PRESS
CHICAGO, ILLINOIS

 PUBLISHED FEBRUARY 1929
Second Edition May 1931

COMPOSED, PRINTED AND BOUND BY THE
GEORGE BANTA PUBLISHING COMPANY
MENASHA, WISCONSIN

THE LAW OF GRAVITATION IN RELATIVITY

CHAPTER I. ELEMENTS OF THE TENSOR THEORY

CHAPTER II. ALGEBRA OF TENSORS

CHAPTER III. DIFFERENTIAL PROPERTIES

CHAPTER IV. THE TENSORS $R_{\rho\mu\nu\lambda}$ AND $G_{\mu\nu}$

CHAPTER V. FUNDAMENTAL TENSORS

CHAPTER VI. THE LAWS OF GRAVITATION

INTRODUCTION

The subject matter of this book falls naturally into two main divisions, an introductory part dealing with tensor analysis, and a second part concerned with the search for all laws of gravitation in empty space admissible under the postulates of Relativity. The problem of deducing these laws from the given set of Relativity postulates is amenable to a purely mathematical treatment, and is one of the most remarkable examples of this sort in modern Physics.

A very powerful weapon in the solution of this problem is the theory of tensors. As originally developed by Ricci and Levi-Civita† this theory was devised to meet a much more general class of geometrical problems than those which later arose in connection with the General Theory of Relativity. In the following treatment we have limited the scope of the theory of tensors in such a way as to lead, in our opinion, to the most fruitful application to Relativity.

As special points in this development we wish to call attention to the following: We commence with a linear differential form, or with sets of such forms, and with matrices defined in a general way. The components of these matrices are explicit functions of the independent variables, their differentials, and the coefficients of one or more linear differential forms. We are then able to set up a simple definition of the transformation of a matrix from one coördinate system to another (§2). Tensors enter the theory as matrices which transform in a specified way (§3). We are then able to show that the components of a tensor do not contain the independent variables explicitly (§3). This procedure makes it possible to establish a general existence theorem providing us with tensors of every order, arbitrary in the coördinates (§9); such a theorem can not be obtained if the theory is based on a quadratic differential form alone. The so-called quotient theorem follows almost at once as a corollary (§10); this can not be proved without a general existence theorem. A precise definition of covariant equations (§6) brings to light the relation of tensors to the expression of laws of nature which satisfy the fundamental postulate of the Theory of Relativity.

The quadratic differential form is introduced (§18) only after the properties of the symmetric covariant tensors of the second order are studied. The fact that the quadratic differential form plays a fundamental rôle in the Theory of Relativity, due to its geometrical significance, does not imply that it must serve as basis of the theory of tensors in Relativity; for every quadratic differential form can be decomposed into the sum of the products of four pairs of linear differential forms.‡

† Ricci and Levi-Civita: *Math. Ann.*, 1901, (54), p. 125.

‡ Or the algebraic sum of the squares of four linear differential forms.

In Chapter V we have been able to give the number and the explicit form of fundamental non-differential tensors of the quadratic differential form. Further, we have shown that every fundamental tensor linear in the second derivatives and containing no higher derivatives, can be expressed as an explicit sum of products of known tensors.

The set of postulates referred to above is given in the opening paragraph of Chapter VI. With the aid of our previous theorems and of three propositions, all the possible forms of the law of gravitation are obtained. They are expressed by four tensor equations (one being Einstein's law), and the combinations of these. Of these four tensors three were previously known, while the fourth, as far as we are aware, is given here for the first time. We have denoted it by $S_{\sigma\tau\rho\mu\nu\lambda}$.

In Chapter VII solutions of these laws are given for the gravitational field of one body. Solutions of the equations of motion of a point-mass in the field of one finite body are given explicitly, including the Einstein law as a special case. These results hold for much more general postulates than those previously considered.

We may state here that all the functions occurring in the following pages shall be assumed to be continuous, and to possess as many continuous derivatives as may be required by the arguments in which they occur. We have not used the double-index summation convention, as we felt that it would be confusing here.

We wish to acknowledge our great debt to the treatments of tensor analysis given by several writers, particularly Einstein,† Eddington‡ and Becquerel.†† As the first five and one half chapters were completed in 1927, at which time we were not familiar with several of the more recent treatments of tensor analysis, such as those of Veblen, Eisenhart, and Birkhoff, we could not take advantage of some of their notational and other innovations. We wish also to express our obligation and gratitude to Professor G. A. Bliss for his encouragement and valuable advice.

† *Annalen d. Physik*, vol. 49, 1916, p. 769.

‡ The Mathematical Theory of Relativity: Cambridge, 1923.

†† *Principe de Relativité et la Théorie de la Gravitation*, Paris, 1922.

CHAPTER I

ELEMENTS OF THE TENSOR THEORY

§1. *The linear differential form.*

We shall deal only with four-dimensional space, and functions of four real variables $x_1, \ldots, x_4$. As the basis of the theory we take the linear differential form

$$dr = \sum_{\mu=1}^{4} h_\mu dx_\mu, \tag{1}$$

where the h_μ are arbitrary functions of $x_1, \ldots, x_4$. Consider the single-valued non-singular transformation of variables

$$\text{(T)} \qquad \begin{cases} x_1 = x_1(x_1', \cdots, x_4') \\ \vdots \\ x_4 = x_4(x_1', \cdots, x_4'). \end{cases}$$

Under this the dx_μ transform as follows:

$$dx_\mu = \sum_{j=1}^{4} \frac{\partial x_\mu}{\partial x_j'} dx_j' \qquad (\mu = 1, \cdots, 4). \tag{2}$$

By substituting (2) in (1) we obtain the transform of (1) under (T):

$$dr = dr' = \sum_{\mu=1}^{4} h_\mu' dx_\mu',$$

where

$$h_\mu'(x_j') \equiv \sum_{i=1}^{4} \frac{\partial x_i}{\partial x_\mu'} h_i[x_j(x_k')] \qquad (\mu = 1, \cdots, 4). \tag{3}$$

The solution of this for h_μ is

$$h_\mu = \sum_{i=1}^{4} \frac{\partial x_i'}{\partial x_\mu} h_i'[x_j'(x_k)] \qquad (\mu = 1, \cdots, 4).$$

In future we shall abbreviate by understanding that all summation indices run from one to four, unless otherwise specified, and that all equations hold for each value from one to four of the free indices.

Since the h_μ of (1) are arbitrary functions of the x_i we may introduce several sets $h_\mu^{(i)}$ of such functions. The product of two such sets $h_\mu^{(1)}\ h_\nu^{(2)}$ may be considered as the coefficients of a symmetric quadratic differential form

$$\sum_{\mu,\nu} h_\mu^{(1)} h_\nu^{(2)} dx_\mu dx_\nu = \sum_{\mu,\nu} h_{\mu\nu} dx_\mu dx_\nu,$$

where

$$h_{\mu\nu} = \tfrac{1}{2}(h_\mu^{(1)} h_\nu^{(2)} + h_\nu^{(1)} h_\mu^{(2)}).$$

§2. *Transformation of matrices.*

We shall deal with matrices of various orders, a matrix of *order* k being an array of 4^k components. For the components we shall admit functions of the quantities $x_1, \ldots, x_4, dx_1, \ldots, dx_4, h_\mu^{(j)}$, and the partial derivatives of all orders of the $h_\mu^{(j)}$ with respect to the x_i.

We shall first consider the simplest type of matrix, namely, one of order zero, a single function $f(x_i, dx_i, h_\mu^{(j)})$; it is understood that every function of the $h_\mu^{(j)}$, such as the one just given, may involve the partial derivatives of the $h_\mu^{(j)}$.

We shall now define the transform of this matrix $f(x_i, dx_i, h_\mu^{(j)})$ in a new system of coördinates when the independent variables x_i are transformed according to (T). Denoting the transform by $f'(x_i, dx_i, h_\mu^{(j)})$ we define it by

$$f'(x_i, dx_i, h_\mu^{(j)}) \equiv f(x_i', dx_i', h_\mu^{(j)\prime}), \tag{4}$$

the $h_\mu^{(j)\prime}$ of the right member being expressed in terms of the x_i' according to the equations (3); in other words the function $f(x_i, dx_i, h_\mu^{(j)})$ is transformed by replacing the quantities $x_i, dx_i, h_\mu^{(j)}$, by $x_i', dx_i', h_\mu^{(j)\prime}$, the partial derivatives $\partial^n h_\mu^{(j)}/\partial x_1^{\alpha_1} \ldots \partial x_4^{\alpha_4}$ $(\sum_i \alpha_i = n)$, being replaced by $\partial^n h_\mu^{(j)\prime}/\partial x_1'^{\alpha_1} \ldots \partial x_4'^{\alpha_4}$.

This law brings to light an important feature of the theory that we are treating. It would be possible to consider the $h_\mu^{(j)}$ in f as functions of the x_i, (which of course they are), so that they would transform by replacing the x_i by the x_i'. But the significance of the present theory is largely dependent upon the explicit presence of the $h_\mu^{(j)}$ in the function f and on their transformation according to (3).

The law of transformation for the general matrix of order k under the transformation of variables (T) is that each component of the matrix transforms separately according to (4). We shall use the notation A' to represent the transform of the matrix A under (T).

The components of A' are functions of the components of A; in case these functions are linear and homogeneous for the general transformation (T), A shall be called a *linear matrix*.

Whenever a quantity is a function of several sets $h_\mu^{(j)}$, we shall abbreviate by writing simply h_μ.

We shall have occasion to use the transform $(\partial A/\partial x_i)'$ of $\partial A/\partial x_i$, for the function $A(x_i, h_i)$. Then

$$\frac{\partial A}{\partial x_i} = A_{x_i}(x_j, h_j) + \sum_k A_{h_k}(x_j, h_j)\frac{\partial h_k}{\partial x_i}.$$

By definition

$$\left(\frac{\partial A}{\partial x_i}\right)' = A_{x_i}(x_j', h_j') + \sum_k A_{h_k}(x_j', h_j')\left(\frac{\partial h_k}{\partial x_i}\right)'.$$

On the other hand

$$\frac{\partial A'}{\partial x_i'} = \frac{\partial}{\partial x_i'}A(x_j', h_j') = A_{x_i}(x_j', h_j') + \sum_k A_{h_k}(x_j', h_j')\frac{\partial h_k'}{\partial x_i'}.$$

By definition

$$\left(\frac{\partial h}{\partial x_i}\right)' = \frac{\partial h'}{\partial x_i'}.$$

Hence

$$\left(\frac{\partial A}{\partial x_i}\right)' = \frac{\partial A'}{\partial x_i'}.$$

§3. *Tensors.*

Among all existing linear matrices there are certain particular ones which we shall call *tensors*. Under this heading we shall include the four following types—*covariant*, *contravariant*, *mixed* and *invariant*, which we now proceed to define. Consider a matrix A of order n with the 4^n components $A(\mu_1, \cdots, \mu_n)$. Under the transformation (T) this becomes a matrix A' of order n, with the components $A'(\mu_1, \cdots, \mu_n)$. If the relations

$$A'(\mu_1, \cdots, \mu_n) = \sum_{\alpha_1, \cdots, \alpha_n} \frac{\partial x_{\alpha_1}}{\partial x'_{\mu}} \cdots \frac{\partial x_{\alpha_n}}{\partial x'_{\mu_n}} A(\alpha_1, \cdots, \alpha_n) \tag{5}$$

are satisfied, the matrix A is called a *covariant tensor*. In this equation $A'(\mu_1, \cdots, \mu_n)$ is expressed according to (4) in terms of x_i, dx_i, h_μ; the partial derivatives $\partial x_{\alpha_i}/\partial x'_{\mu_j}$ are expressed in terms of the x_i by (T). By the equation we mean an identity in the x_i, the dx_i, the h_μ and their derivatives. Every equation involving the x_i ,dx_i, and the h_μ of two coordinate systems has precisely this meaning.

If the relations

$$A'(\nu_1, \cdots, \nu_n) = \sum_{\beta_1, \cdots, \beta_n} \frac{\partial x'_{\nu_1}}{\partial x_{\beta_1}} \cdots \frac{\partial x'_{\nu_n}}{\partial x_{\beta_n}} A(\beta_1, \cdots, \beta_n) \tag{6}$$

are satisfied, A is called a *contravariant tensor*.

If the relations

$$(7)\quad A'(\mu_1, \cdots, \mu_k; \nu_1, \cdots, \nu_l) = \sum_{\substack{\alpha_1, \cdots, \alpha_k \\ \beta_1, \cdots, \beta_l}} \frac{\partial x_{\alpha_1}}{\partial x'_{\mu_1}} \cdots \frac{\partial x_{\alpha_k}}{\partial x'_{\mu_k}} \frac{\partial x'_{\nu_1}}{\partial x_{\beta_1}} \cdots \frac{\partial x'_{\nu_l}}{\partial x_{\beta_l}} A(\alpha_1, \cdots, \alpha_k; \beta_1, \cdots, \beta_l)$$

are satisfied, A is called a *mixed tensor*, covariant of order k, contravariant of order l. The solution of (7) is

$$(8)\quad A(\mu_1, \cdots, \mu_k; \nu_1, \cdots, \nu_l) = \sum_{\substack{\alpha_1, \cdots, \alpha_k \\ \beta_1, \cdots, \beta_l}} \frac{\partial x'_{\alpha_1}}{\partial x_{\mu_1}} \cdots \frac{\partial x'_{\alpha_k}}{\partial x_{\mu_k}} \frac{\partial x_{\nu_1}}{\partial x'_{\beta_1}} \cdots \frac{\partial x_{\nu_l}}{\partial x'_{\beta_l}} A'(\alpha_1, \cdots, \alpha_k; \beta_1, \cdots, \beta_l).$$

If the relations

$$(9)\qquad A'(\mu_1, \cdots, \mu_n) = A(\mu_1, \cdots, \mu_n)$$

are satisfied, the matrix A is called an *invariant matrix*. If A is also a tensor it is an *invariant tensor*.

We shall hereafter use the term *valence* for the *order* of a tensor.†

The tensors (5), (6), and (7) are represented by the following notation:

The covariant tensor (5) is $A_{\mu_1, \cdots, \mu_n}$ or simply $A_{(n)}$ or $A_{(n)}^{(0)}$.

The contravariant tensor (6) is $A^{\nu_1, \cdots, \nu_n}$, or simply $A^{(n)}$ or $A_{(0)}^{(n)}$.

The mixed tensor (7) is $A_{\mu_1, \cdots, \mu_k}^{\nu_1, \cdots, \nu_l}$, or simply $A_{(k)}^{(l)}$.

As implied above, the term mixed tensor will be used to include a tensor $A_{(n)} = A_{\mu_1, \cdots, \mu_n}$, covariant of valence n, contravariant of valence zero, a tensor $A^{(n)} = A^{\nu_1, \cdots, \nu_n}$, contravariant of valence n, covariant of valence zero, and an invariant tensor $A_{(0)}^{(0)}$, covariant of valence zero, contravariant of valence zero, a *single* invariant function.

The notation $A_{(k)}^{(l)}$ shows that A is a mixed tensor of valence $k+l$. For the components of this tensor it is convenient to use the notation $A(\mu_1, \cdots, \mu_k; \nu_1, \cdots, \nu_l)$ or $A_{(k)}^{(l)}\ (\mu_1, \cdots, \mu_k; \nu_1, \cdots, \nu_l)$. The tensor $A_{(k)}^{(l)}$ is called a tensor of *type* $\binom{l}{k}$.

Since the tensor property as defined by (7) is a property of the function A and not of its parameters, it follows that if $A(x_i, dx_i, h_\mu)$ is a tensor of type $\binom{l}{k}$ then $A(x_i', dx_i', h_\mu')$ is also a tensor of the same type; hence the tensor property is independent of the coördinate system. This also follows directly: By (4) we see that A is the transform of A' under (T^{-1}); thus if A is a tensor then (8) is true for all (T^{-1}), whence by definition A' is a tensor of the same type as A.

† Due to Professor A. C. Lunn.

THEOREM. *A tensor cannot involve x_i explicitly.*

Let $A_{(k)}^{(l)}$ be a tensor of type $\binom{l}{k}$, the components being functions of x_i, dx_i, h_μ. Make the transformation $x_i = x_i' - a_i$, where the a_i are constants. Then $\partial x_i/\partial x_j' = I(i,j)$, I being the identity matrix. Then by (7)

$$A'(\mu_1, \cdots, \mu_k; \nu_1, \cdots, \nu_l) = \sum_{\substack{\alpha_1, \cdots, \alpha_k \\ \beta_1, \cdots, \beta_l}} I(\alpha_1,\mu_1) \cdots I(\alpha_k,\mu_k) I(\beta_1,\nu_1) \cdots$$

$$\cdots I(\beta_l,\nu_l) A(\alpha_1, \cdots, \alpha_k; \beta_1, \cdots, \beta_l) = A(\mu_1, \cdots, \mu_k; \nu_1, \cdots, \nu_l).$$

Hence for each component $f(x_i, dx_i, h_\mu)$ of $A_{(k)}^{(l)}$ we have

$$f'(x_i, dx_i, h_\mu) = f(x_i, dx_i, h_\mu).$$

By definition $f'(x_i, dx_i, h_\mu) = f(x_i', dx_i', h_\mu')$. Since $x_i' = x_i + a_i$ we have

$$dx_i' = dx_i,$$

and by (3)

$$h_\mu'(x') = h_\mu(x).$$

Now

$$\frac{\partial h_\mu'}{\partial x_\sigma'} = \frac{\partial h_\mu}{\partial x_\sigma'} = \sum_k \frac{\partial h_\mu}{\partial x_k} \frac{\partial x_k}{\partial x_\sigma'} = \sum_k \frac{\partial h_\mu}{\partial x_k} I(k,\sigma) = \frac{\partial h_\mu}{\partial x_\sigma}.$$

Similarly

$$\frac{\partial^n h_\mu'}{\partial x_1'^{\alpha_1} \cdots \partial x_4'^{\alpha_4}} = \frac{\partial^n h_\mu}{\partial x_1^{\alpha_1} \cdots \partial x_4^{\alpha_4}}.$$

Hence

$$f(x_i', dx_i', h_\mu') = f(x_i + a_i, dx_i, h_\mu).$$

Thus

$$f(x_i + a_i, dx_i, h_\mu) = f(x_i, dx_i, h_\mu).$$

Since the a_i are arbitrary constants it follows that f does not involve x_i explicitly; hence the components of $A_{(k)}^{(l)}$ are functions of dx_i, h_μ alone.

§4. *Lemmas.*

For future reference we introduce at this point a few lemmas.

Lemma I.

$$\sum_\gamma \frac{\partial x_\alpha}{\partial x_\gamma'} \frac{\partial x_\gamma'}{\partial x''_\beta} = \frac{\partial x_\alpha}{\partial x''_\beta}.$$

This follows directly from the usual law of multiplication of matrices applied to

$$\left(\frac{\partial x_i}{\partial x_j'}\right) \text{ and } \left(\frac{\partial x_i'}{\partial x_j''}\right),$$

using the well-known theorem on Jacobians.†

Lemma II.

$$\sum_{\gamma} \frac{\partial x_\alpha}{\partial x_\gamma'} \frac{\partial x_\gamma'}{\partial x_\beta} = \frac{\partial x_\alpha}{\partial x_\beta} = I(\alpha,\beta).$$

This follows immediately from Lemma I by putting $x''=x$, or if preferred, by multiplying together the matrices

$$\left(\frac{\partial x_i}{\partial x_j'}\right) \text{ and } \left(\frac{\partial x_i'}{\partial x_j}\right).$$

Lemma III.

$$\sum_{\beta,\gamma} \frac{\partial x_\alpha}{\partial x_\gamma'} \frac{\partial x_\gamma'}{\partial x_\beta} A(\alpha,\beta,\rho_1, \cdots ,\rho_r) = A(\alpha,\alpha,\rho_1, \cdots ,\rho_r),$$

where A is any matrix of order $r+2$. This follows at once from Lemma II.

§5. *The group property of tensors.*

The tensors as defined in §3 satisfy the group property. By this is meant the following: consider two arbitrary non-singular transformations of variables

$$\text{(T)} \qquad x_i = x_i(x_j')$$

and

$$\text{(T}') \qquad x_j' = x_j'(x_l'').$$

If a matrix A transforms under (T) to a matrix A' according to

$$(10) \qquad A'(\rho_1, \cdots ,\rho_n) = \sum_{\alpha_1, \cdots ,\alpha_n} \frac{\partial x_{\alpha_1}}{\partial x_{\rho_1}'} \cdots \frac{\partial x_{\alpha_k}}{\partial x_{\rho_k}'} \frac{\partial x_{\rho_{k+1}}'}{\partial x_{\alpha_{k+1}}} \cdots \frac{\partial x_{\rho_n}'}{\partial x_{\alpha_n}} A(\alpha_1, \cdots ,\alpha_n),$$

and the matrix A' transforms under (T′) to a matrix A'' according to

† Goursat-Hedrick: Mathematical Analysis, vol. I, § 29.

$$(11) \qquad A''(\rho_1, \cdots, \rho_n) = \sum_{\beta_1, \cdots, \beta_n} \frac{\partial x'_{\beta_1}}{\partial x''_{\rho_1}} \cdots \frac{\partial x'_{\beta_k}}{\partial x''_{\rho_k}} \frac{\partial x''_{\rho_{k+1}}}{\partial x'_{\beta_{k+1}}} \cdots \frac{\partial x''_{\rho_n}}{\partial x'_{\beta_n}} A'(\beta_1, \cdots, \beta_n),$$

then under the composite transformation

$$(\mathrm{T'T}) \qquad x_i = x_i[x'_j(x''_l)]$$

the matrix A transforms into the matrix A'' according to

$$(12) \qquad A''(\rho_1, \cdots, \rho_n) = \sum_{\gamma_1, \cdots, \gamma_n} \frac{\partial x_{\gamma_1}}{\partial x''_{\rho_1}} \cdots \frac{\partial x_{\gamma_k}}{\partial x''_{\rho_k}} \frac{\partial x''_{\rho_{k+1}}}{\partial x_{\gamma_{k+1}}} \cdots \frac{\partial x''_{\rho_n}}{\partial x_{\gamma_n}} A(\gamma_1, \cdots, \gamma_n).$$

To prove the theorem it is merely necessary to substitute the A' of (10) in (11), and to note that by Lemma I

$$\sum_{\beta} \frac{\partial x'_\beta}{\partial x''_\rho} \frac{\partial x_\alpha}{\partial x'_\beta} = \frac{\partial x_\alpha}{\partial x''_\rho}.$$

Since, as just shown, mixed tensors satisfy the group property, it follows that all tensors satisfy it.

§6. *Covariant equations.*

For the sake of clarity it seems well to state at this point the object of the present analysis. This object is to find covariant equations of the linear differential forms

$$(1)' \qquad dr^{(j)} = \sum_{\mu} h_\mu^{(j)} dx_\mu.$$

By a covariant equation of $(1)'$ is meant an equation

$$(13) \qquad f(x_i, dx_i, h_\mu) = 0$$

such that, for every transformation (T), the equation

$$f(x_i', dx_i', h_\mu') = 0$$

is satisfied whenever (13) is satisfied. It is to be noted that according to (4) this condition is that the transform f' of f satisfies

$$f'(x_i, dx_i, h_\mu) = 0$$

whenever (13) is satisfied.

By a set of covariant equations is meant a set

$$f_\alpha(x_i, dx_i, h_\mu) = 0 \qquad (\alpha = 1, \cdots, n)$$

such that, for every transformation (T), the set of equations

$$f_\alpha(x_i', dx_i', h_\mu') = 0 \qquad (\alpha = 1, \cdots, n)$$

is satisfied whenever the first set is satisfied. Again it is to be noted that this condition is that all the transforms f_α' of the f_α vanish whenever all the f_α vanish.

We shall now show how the theory of tensors, as developed in §§ 1 to 5, permits us to find examples of covariant equations.

By the equation

$$A = 0,$$

A being a matrix, is meant the set of equations

$$A(\mu_1, \cdots, \mu_n) = 0 .$$

If A is a tensor, then $A = 0$ implies that $A' = 0$ under all transformations (T). Hence for every tensor A the set of equations $A = 0$ is a set of covariant equations of (1)′. The search for examples of sets of covariant equations may thus be conducted by searching for tensors. It must be noted, however, that *we have not shown whether or not there exist covariant equations not equivalent to tensor equations.*

CHAPTER II

ALGEBRA OF TENSORS

§7 *Algebraic properties of tensors.*

We proceed to consider some algebraic properties of tensors. The *sum* or *difference* $A \pm B$ of two tensors $A_{(k)}^{(l)}$, $B_{(k)}^{(l)}$ is defined as C where

$$C(\mu_1, \cdots, \mu_k; \nu_1, \cdots, \nu_l) \equiv A(\mu_1, \cdots, \mu_k; \nu_1, \cdots, \nu_l) \pm B(\mu_1, \cdots, \mu_k; \nu_1, \cdots, \nu_l).$$

It follows immediately from the linearity of (7) that C is a tensor $C_{(k)}^{(l)}$.

The product AB of any two tensors $A_{(m)}^{(n)}$, $B_{(s)}^{(t)}$ is defined as C where

$$C(\mu_1, \cdots, \mu_m; \nu_1, \cdots, \nu_n; \sigma_1 \cdots, \sigma_s; \tau_1, \cdots, \tau_t) \equiv A(\mu_1, \cdots, \mu_m; \nu_1, \cdots, \nu_n) B(\sigma_1, \cdots, \sigma_s; \tau_1, \cdots, \tau_t).$$

That C is a tensor $C_{(m+s)}^{(n+t)}$ follows readily from (7). For example, consider $C = A_{(1)}^{(1)} B_{(1)}$. By definition

$$A'(\mu,\nu) = \sum_{\alpha,\beta} \frac{\partial x_\alpha}{\partial x'_\mu} \frac{\partial x'_\nu}{\partial x_\beta} A(\alpha,\beta); \quad B'(\sigma) = \sum_\gamma \frac{\partial x_\gamma}{\partial x'_\sigma} B(\gamma).$$

Hence

$$A'(\mu,\nu)B'(\sigma) = \sum_{\alpha,\beta,\gamma} \frac{\partial x_\alpha}{\partial x'_\mu} \frac{\partial x_\gamma}{\partial x'_\sigma} \frac{\partial x'_\nu}{\partial x_\beta} A(\alpha,\beta)B(\gamma).$$

Since

$$C'(\mu,\nu,\sigma) = A'(\mu,\nu)B'(\sigma) \quad \text{and} \quad C(\alpha,\beta,\gamma) = A(\alpha,\beta)B(\gamma)$$

it follows that

$$C'(\mu,\nu,\sigma) = \sum_{\alpha,\beta,\gamma} \frac{\partial x_\alpha}{\partial x'_\mu} \frac{\partial x_\gamma}{\partial x'_\sigma} \frac{\partial x'_\nu}{\partial x_\beta} C(\alpha,\beta,\gamma),$$

which shows that C is a tensor $C_{(2)}^{(1)}$. The product of two tensors as defined above is called the *outer product.*

We shall now define the term *contracted tensor.* For a tensor $A_{(k)}^{(l)}$, $k+l>0$, consider the matrix C defined by

$$C(\mu_1, \cdots, \mu_{k-1}; \nu_1, \cdots, \nu_{l-1}) \equiv \sum_\rho A(\mu_1, \cdots, \mu_{k-1}, \rho; \nu_1, \cdots, \nu_{l-1}, \rho).$$

It is readily shown that C is a tensor of type $\binom{l-1}{k-1}$. For

$$C'(\mu_1, \cdots, \mu_{k-1}; \nu_1, \cdots, \nu_{l-1}) = \sum_{\rho} A'(\mu_1, \cdots, \mu_{k-1}, \rho; \nu_1, \cdots, \nu_{l-1}, \rho)$$

$$= \sum_{\rho} \sum_{\substack{\alpha_1, \cdots, \alpha_k \\ \beta_1, \cdots, \beta_l}} \frac{\partial x_{\alpha_1}}{\partial x'_{\mu_1}} \cdots \frac{\partial x_{\alpha_{k-1}}}{\partial x'_{\mu\,k-1}} \frac{\partial x_{\alpha_k}}{\partial x'_{\rho}} \frac{\partial x'_{\nu_1}}{\partial x_{\beta_1}} \cdots$$

$$\frac{\partial x'_{\nu\,l-1}}{\partial x_{\beta_{l-1}}} \frac{\partial x'_{\rho}}{\partial x_{\beta_l}} A(\alpha_1, \cdots, \alpha_{k-1}, \alpha_k; \beta_1, \cdots, \beta_{l-1}.\beta_l)$$

$$= \sum_{\substack{\alpha_1, \cdots, \alpha_k \\ \beta_1, \cdots, \beta_{l-1}}} \frac{\partial x_{\alpha_1}}{\partial x'_{\mu_1}} \cdots \frac{\partial x_{\alpha_{k-1}}}{\partial x'_{\mu\,k-1}} \frac{\partial x'_{\nu_1}}{\partial x_{\beta_1}} \cdots \frac{\partial x'_{\nu\,l-1}}{\partial x_{\beta_{l-1}}}$$

$$\sum_{\beta_l, \rho} \frac{\partial x_{\alpha_k}}{\partial x'_{\rho}} \frac{\partial x'_{\rho}}{\partial x_{\beta_l}} A(\alpha_1, \cdots, \alpha_{k-1}, \alpha_k; \beta_1, \cdots, \beta_{l-1}, \beta_l).$$

By Lemma III of §4

$$\sum_{\beta_l, \rho} \frac{\partial x_{\alpha_k}}{\partial x'_{\rho}} \frac{\partial x'_{\rho}}{\partial x_{\beta_l}} A(\alpha_1, \cdots, \alpha_{k-1}, \alpha_k; \beta_1, \cdots, \beta_{l-1}, \beta_l)$$

$$= A(\alpha_1, \cdots, \alpha_{k-1}, \alpha_k; \beta_1, \cdots, \beta_{l-1}, \alpha_k).$$

Hence, setting $\alpha_k = \sigma$

$$C'(\mu_1, \cdots, \mu_{k-1}; \nu_1, \cdots, \nu_{l-1}) = \sum_{\substack{\alpha_1, \cdots, \alpha_{k-1} \\ \beta_1, \cdots, \beta_{l-1}}} \frac{\partial x_{\alpha_1}}{\partial x'_{\mu_1}} \cdots \frac{\partial x_{\alpha_{k-1}}}{\partial x'_{\mu\,k-1}} \frac{\partial x'_{\nu_1}}{\partial x_{\beta_1}}$$

$$\cdots \frac{\partial x'_{\nu\,l-1}}{\partial x_{\beta_{l-1}}} \sum_{\sigma} A(\alpha_1, \cdots, \alpha_{k-1}, \sigma; \beta_1, \cdots, \beta_{l-1}, \sigma)$$

$$= \sum_{\substack{\alpha_1 \cdots, \alpha_{k-1} \\ \beta_1 \cdots, \beta_{l-1}}} \frac{\partial x_{\alpha_1}}{\partial x'_{\mu}} \cdots \frac{\partial x_{\alpha_{k-1}}}{\partial x'_{\mu_{k-1}}} \frac{\partial x'_{\nu_1}}{\partial x_{\beta_1}} \cdots$$

$$\frac{\partial x'_{\nu\,l-1}}{\partial x_{\beta_{l-1}}} C(\alpha_1, \cdots, \alpha_{k-1}; \beta_1, \cdots, \beta_{l-1}),$$

which shows that C is a tensor $C\binom{l-1}{k-1}$. It is a *once contracted tensor* of $A\binom{l}{k}$. This process is called *contraction*, with respect to μ_k, ν_l. Contraction with respect to two indices μ_1, μ_2 or ν_1, ν_2 of the same character, does not yield

a tensor, and is of no interest to us. Therefore the term contraction will always imply that the indices are of opposite character, that is, one of covariant, and one of contravariant character. It is evident that we also obtain a tensor of type $\binom{l-1}{k-1}$ by contraction with respect to any of the kl pairs μ_i, ν_j. It is an immediate consequence of the theorem just proved that by contracting a tensor $A\binom{(k)}{(k)}$ successively k times we obtain a tensor $C\binom{(0)}{(0)}$, which is an invariant.

We shall now define the term *inner product* of two tensors $A\binom{(n)}{(m)}$, $B\binom{(m)}{(n)}$. If we contract the tensor $C\binom{(n+m)}{(m+n)} = A\binom{(n)}{(m)} B\binom{(m)}{(n)}$ successively $m+n$ times we obtain an invariant $C\binom{(0)}{(0)}$, an inner product of A and B. Since it is possible to choose the pairs of indices with respect to which the successive contractions are made in arbitrary fashion there exists more than one inner product of A and B; in fact there are $(m+n)!$ such inner products, all of which may be distinct.

Consider two matrices A and B, of order 2. We shall denote by (AB) the matrix obtained from A and B by ordinary matricial multiplication. Let A', B', and $(AB)'$ be the transforms of A, B, and (AB) respectively. It follows that

$$(AB)' = A'B'.$$

For, let

$$A = \begin{Vmatrix} a_{11}, & \cdots, & a_{14} \\ \vdots & & \vdots \\ a_{41}, & \cdots, & a_{44} \end{Vmatrix}, \quad B = \begin{Vmatrix} b_{11}, & \cdots, & b_{14} \\ \vdots & & \vdots \\ b_{41}, & \cdots, & b_{44} \end{Vmatrix}.$$

Then

$$(AB) = \begin{Vmatrix} \sum_i a_{1i}b_{i1}, & \cdots, & \sum_i a_{1i}b_{i4} \\ \vdots & & \vdots \\ \sum_i a_{4i}b_{i1}, & \cdots, & \sum_i a_{4i}b_{i4} \end{Vmatrix}.$$

According to the law of transformation of a matrix (§ 2)

$$A' = \begin{Vmatrix} a_{11}', & \cdots, & a_{14}' \\ \vdots & & \vdots \\ a_{41}', & \cdots, & a_{44}' \end{Vmatrix}, \quad B' = \begin{Vmatrix} b_{11}', & \cdots, & b_{14}' \\ \vdots & & \vdots \\ b_{41}', & \cdots, & b_{44}' \end{Vmatrix}$$

where a_{ij}' is the transform of a_{ij}, and b_{ij}' is the transform of b_{ij}. Hence

$$(A'B') = \begin{Vmatrix} \sum_i a_{1i}' b_{i1}', & \cdots, & \sum_i a_{1i}' b_{i4}' \\ \vdots & & \vdots \\ \sum_i a_{4i}' b_{i1}', & \cdots, & \sum_i a_{4i}' b_{i4}' \end{Vmatrix}$$

Also

$$(AB)' = \left\| \begin{array}{ccc} \left(\sum_i a_{1i}b_{i1}\right)', & \cdots, & \left(\sum_i a_{1i}b_{i4}\right)' \\ \vdots & & \vdots \\ \left(\sum_i a_{4i}b_{i1}\right)', & \cdots, & \left(\sum_i a_{4i}b_{i4}\right)' \end{array} \right\|.$$

Since

$$\left(\sum_i a_{ji}b_{ik}\right)' = \sum_i a_{ji}'b_{ik}',$$

as is apparent from the law (4) for the transformation of a function, it follows at once that $(AB)' = (A'B')$.

The matrix A is said to be *symmetric* in ρ_i, ρ_j if

$$A(\rho_1, \cdots, \rho_i, \cdots, \rho_j, \cdots, \rho_n) = A(\rho_1, \cdots, \rho_j, \cdots, \rho_i, \cdots, \rho_n).$$

The matrix A is said to be *antisymmetric* in ρ_i, ρ_j if

$$A(\rho_1, \cdots, \rho_i, \cdots, \rho_j, \cdots, \rho_n) = -A(\rho_1, \cdots, \rho_j, \cdots, \rho_i, \cdots, \rho_n).$$

The matrix A is said to be *completely symmetric* or *completely antisymmetric* in case it is symmetric or antisymmetric respectively in every pair ρ_i, ρ_j.

It follows at once from the definition of transform that if A is a symmetric or an antisymmetric matrix, then A' is a symmetric or an antisymmetric matrix, respectively. For if $A(\alpha, \beta, \rho_j) = A(\beta, \alpha, \rho_j)$ it follows from (4) that the A' satisfy the same relations. These definitions of course apply in the particular case where A is a tensor.

§8. *Reciprocal of* $A_{(2)}$.

Consider a particular covariant tensor of valence two, $A_{\mu\nu}$, with det $A \neq 0$. Let $B = A^{-1}$ represent the reciprocal of the matrix A; then $(AB) = I$. We shall show that B is a contravariant tensor of valence two.

By § 7, $(AB)' = (A'B')$. But $(AB)' = I' = I$. Therefore $(A'B') = I$. We have

$$(A'B')(\mu,\nu) = \sum_i A'_{\mu i}B'(i,\nu) = I(\mu,\nu).$$

By (5)

$$A'_{\mu i} = \sum_{\alpha,\beta} \frac{\partial x_\alpha}{\partial x_\mu'} \frac{\partial x_\beta}{\partial x_i'} A_{\alpha\beta}.$$

Hence

$$\sum_{i,\alpha,\beta} \frac{\partial x_\alpha}{\partial x_\mu'} \frac{\partial x_\beta}{\partial x_i'} A_{\alpha\beta}B'(i,\nu) = I(\mu,\nu).$$

Multiply each side of this equation by

$$\frac{\partial x'_\rho}{\partial x_\delta} B(\delta,\gamma) \frac{\partial x'_\mu}{\partial x_\gamma}$$

and sum as to δ, γ, μ. We then have

$$\sum_i B'(i,\nu) \sum_\delta \frac{\partial x'_\rho}{\partial x_\delta} \sum_\beta \frac{\partial x_\beta}{\partial x'_i} \sum_\gamma B(\delta,\gamma) \sum_{\alpha,\mu} \frac{\partial x_\alpha}{\partial x'_\mu} \frac{\partial x'_\mu}{\partial x_\gamma} A_{\alpha\beta}$$
$$= \sum_{\delta,\gamma} \frac{\partial x'_\rho}{\partial x_\delta} B(\delta,\gamma) \sum_\mu \frac{\partial x'_\mu}{\partial x_\gamma} I(\mu,\nu).$$

By Lemmas II and III of § 4 this becomes

$$B'(\rho,\nu) = \sum_{\delta,\gamma} \frac{\partial x'_\rho}{\partial x_\delta} \frac{\partial x'_\nu}{\partial x_\gamma} B(\delta,\gamma),$$

which shows, by (6), that $B = A^{-1}$ is a contravariant tensor.

It is interesting to note that, with the commonly accepted definition, the reciprocal of the matrix h_μ of (1) is not a contravariant tensor as might be expected from the results of this section. For the reciprocal in question is

$$h_\mu^{-1} = \frac{h_\mu}{\sum_k h_k^2}.$$

That this is not a contravariant tensor is seen by applying the transformation

$$\begin{aligned} x'_1 &= x_1 + x_2 \\ x'_2 &= x_1 - x_2 \\ x'_3 &= x_3 \\ x'_4 &= x_4 . \end{aligned}$$

§9. *Existence theorem.*

In preceding sections we have defined tensors in a general way, and have discussed some of their algebraic properties. The question naturally arises as to the existence of tensors of various *types*. In answer to this question we shall state and prove the following theorem:

For every pair of indices m, n, there exists a general tensor of type $\binom{n}{m}$, *a function of* $h_\mu^{(i)}$.

By a *general* tensor of type $\binom{n}{m}$ we mean a tensor $A_{\binom{n}{m}}$ whose components as functions of the x_i are arbitrary except for the condition that the tensor is non-degenerate, that is, that for every choice of $m+n-1$ indices the four components corresponding to the remaining index are not all identically zero.

We shall first consider the case where $n=0$, $m\neq 0$. For $m=1$ the theorem is obviously true, for the covariant tensor h_μ of (1) is arbitrary in the x_i by definition. For the general m we proceed by induction. Suppose the theorem true for $m-1$, that is, that there exists a general tensor $A_{(m-1)}$. In proof consider the expression

$$A_{\mu_1,\cdots,\mu_{m-1}}M_{\mu_m}+\cdots+D_{\mu_1,\cdots,\mu_{m-1}}Q_{\mu_m}, \tag{14}$$

where A, B, C, D are general tensors of type $\binom{0}{m-1}$, and M, N, P, Q general tensors of type $\binom{0}{1}$. It follows from § 7 that this expression is a tensor of type $\binom{0}{m}$. We wish now to show that this tensor (14) is general, that is, that we can choose A, B, C, D, M, N, P, Q such that the components of (14) are arbitrary functions (except for the condition mentioned) of x_i, say $T(\mu_1, \cdots, \mu_m)$. To this end let us choose the arbitrary components $A, \cdots, Q$ in the particular coördinate system in which the $T(\mu_1, \cdots, \mu_m)$ are given, as follows:

$$M = 1,0,0,0$$

$$N = 0,1,0,0$$

$$P = 0,0,1,0$$

$$Q = 0,0,0,1$$

$$A_{\mu_1,\cdots,\mu_{m-1}} = T(\mu_1, \cdots, \mu_{m-1}, 1)$$

$$B_{\mu_1,\cdots,\mu_{m-1}} = T(\mu_1, \cdots, \mu_{m-1}, 2)$$

$$C_{\mu_1,\cdots,\mu_{m-1}} = T(\mu_1, \cdots, \mu_{m-1}, 3)$$

$$D_{\mu_1,\cdots,\mu_{m-1}} = T(\mu_1, \cdots, \mu_{m-1}, 4)\ .$$

Then the components of (14) are $T(\mu_1, \cdots, \mu_m)$, as was to be shown. To complete the proof it is necessary to note that since by hypothesis the tensors $A, \cdots, Q$ are non-degenerate, the particular choice we have made is legitimate.

The proof of the theorem for $m=0$, $n\neq 0$ is completely analogous to the proof for $n=0$, $m\neq 0$, after we establish the existence of a general contravariant tensor of valence one, which we proceed to do, first introducing the following lemma.

Lemma: There exists a contravariant tensor of valence two, $B^{\mu\nu}$ whose components are functions of the x_i, arbitrary except for the condition det $B\neq 0$.

For let T be a matrix of the second order whose components are functions of the x_i, arbitrary except that det $T\neq 0$; then det $T^{-1}\neq 0$. By the theorem for $n=0$, $m\neq 0$ there exists a covariant tensor $A_{\mu\nu}=T^{-1}$. Now by §8 we know that there exists a contravariant tensor $B^{\mu\nu}=A^{-1}$. But $A^{-1}=T$; hence $B^{\mu\nu}=T$, which proves the lemma.

Now consider the expression

$$\sum_{\nu} [A^{\mu\nu}M_{\nu} + B^{\mu\nu}N_{\nu} + C^{\mu\nu}P_{\nu} + D^{\mu\nu}Q_{\nu}], \tag{15}$$

where A, B, C, D are tensors of type $\binom{2}{0}$, arbitrary except that their determinants are all not identically zero, and M, N, P, Q are general tensors of type $\binom{0}{1}$. It follows from §7 that this expression is a tensor of type $\binom{1}{0}$. We wish now to show that this tensor (15) is general, that is, that we can choose $A, \cdots, Q$ such that the components of (15) are functions of the x_i, say $T(\mu)$, arbitrary except that not all are identically zero. To this end let us choose the arbitrary components of M, N, P, Q as follows:

$$M = 1,0,0,0$$

$$N = 0,1,0,0$$

$$P = 0,0,1,0$$

$$Q = 0,0,0,1.$$

Then (15) becomes

$$A^{\mu 1} + B^{\mu 2} + C^{\mu 3} + D^{\mu 4}. \tag{15$'$}$$

Since not all the components $T(\mu)$ are zero, suppose that $T(1)\neq 0$. Now choose A, B, C, D as follows, in the same coördinate system:

$$A^{\mu 1} = T(1),0,0,0$$

$$B^{\mu 2} = 0,T(2),T(1),0$$

$$C^{\mu 3} = 0,T(1),T(3),0$$

$$D^{\mu 4} = 0,-T(1),-T(1),T(4).$$

Then the components of (15)$'$, and hence of (15), are $T(\mu)$, as was to be shown. To complete the proof it is necessary to note that our choice of

M, N, P, Q and of A, B, C, D is legitimate, for we can easily choose $A^{\mu\nu}$ $(\nu\neq 1)$, $B^{\mu\nu}(\nu\neq 2)$, $C^{\mu\nu}(\nu\neq 3)$, $D^{\mu\nu}$ $(\nu\neq 4)$ so that det $A\neq 0$, det $B\neq 0$, det $C\neq 0$, det $D\neq 0$.

We have thus proved the existence of a general contravariant tensor of valence one, and hence according to our previous statement the theorem is true for $m=0$, $n\neq 0$.

If $m\neq 0$, and also $n\neq 0$, the theorem is proved in exactly the same manner. For we first obtain a general tensor of type $\binom{1}{m}$ as follows:

$$A_{(m)}^{(1)} = A_{(m)}M^{(1)} + \cdots + D_{(m)}Q^{(1)} ;$$

next a general tensor of type $\binom{2}{m}$

$$A_{(m)}^{(2)} = A_{(m)}^{(1)} \quad M^{(1)} + \cdots + D_{(m)}^{(1)} \quad Q^{(1)} ,$$

and so on.

To prove the theorem for the remaining case $(m, n)=(0, 0)$ we consider the expression

$$\sum_{\mu} A^{\mu}B_{\mu},$$

where A, B are general tensors of type indicated. This expression is a tensor of type $\binom{0}{0}$, i. e. an invariant, by §7. Choose

$$A = 1,0,0,0$$

$$B = T,1,0,0$$

where T is arbitrary. Then $\sum_{\mu} A^{\mu}B_{\mu}$ has the value T, which is arbitrary. Hence we have an arbitrary tensor $A\left\{{0 \atop 0}\right\}$ of valence zero.

Corollary. For every pair of indices m, n, there exists a tensor $A\left\{{n \atop m}\right\}$ such that $A'\left\{{n \atop m}\right\}$ is a general tensor.

For consider the general tensor $B\left\{{n \atop m}\right\}$, and let $A\left\{{n \atop m}\right\}$ be the tensor obtained from $B\left\{{n \atop m}\right\}$ by the transformation (T^{-1}); then $B\left\{{n \atop m}\right\}=A'\left\{{n \atop m}\right\}$ which shows that $A'\left\{{n \atop m}\right\}$ is a general tensor.

§10. *Quotient theorem.*

We have seen in §7 that

$$\sum_{\substack{\mu_1,\cdots,\mu_m \\ \nu_1,\cdots,\nu_n}} A_{(m+p)}^{(n+r)} (\mu_1, \cdots, \mu_m, \pi_1, \cdots, \pi_p ; \nu_1, \cdots, \nu_n, \rho_1, \cdots, \rho_r)$$

$$B_{(n+s)}^{(m+t)} (\nu_1, \cdots, \nu_n, \sigma_1, \cdots, \sigma_s ; \mu_1, \cdots, \mu_m, \tau_1, \cdots, \tau_t) = C_{(p+s)}^{(r+t)}.$$

We wish now to prove a *quotient theorem:*
If for every tensor $A^{(n+r)}_{(m+p)}$

$$\sum_{\substack{\mu_1,\cdots,\mu_m \\ \nu_1,\cdots,\nu_n}} A^{(n+r)}_{(m+p)}(\mu_1,\cdots,\mu_m,\pi_1,\cdots,\pi_p;\nu_1,\cdots,\nu_n,\rho_1,\cdots,\rho_r)$$

$$B(\nu_1,\cdots,\nu_n,\sigma_1,\cdots,\sigma_s;\mu_1,\cdots,\mu_m,\tau_1,\cdots,\tau_t) = C(\pi,\sigma;\rho,\tau)$$

is a tensor of type $\binom{r+t}{p+s}$ *then* B *is a tensor of type* $\binom{m+t}{n+s}$. In the proof we shall let $\sum_\alpha \partial x_\alpha/\partial x'_\mu$ represent $\sum_{\alpha_1,\cdots,\alpha_m} \partial x_{\alpha_1}/\partial x'_{\mu_1}\cdots\partial x_{\alpha_m}/\partial x'_{\mu_m}$, etc. By hypothesis

$$C'(\pi,\sigma;\rho,\tau) = \sum_{\theta,\kappa,\iota,\lambda}\frac{\partial x_\theta}{\partial x'_\pi}\frac{\partial x_\kappa}{\partial x'_\sigma}\frac{\partial x'_\rho}{\partial x_\iota}\frac{\partial x'_\tau}{\partial x_\lambda}C(\theta,\kappa;\iota,\lambda); \qquad \text{that is}$$

$$\sum_{\mu,\nu} A'(\mu,\pi;\nu,\rho)B'(\nu,\sigma;\mu,\tau)$$

$$= \sum_{\theta,\iota,\kappa,\lambda,\mu,\nu}\frac{\partial x_\theta}{\partial x'_\pi}\frac{\partial x_\kappa}{\partial x'_\sigma}\frac{\partial x'_\rho}{\partial x_\iota}\frac{\partial x'_\tau}{\partial x_\lambda}A(\mu,\theta;\nu,\iota)B(\nu,\kappa;\mu,\lambda)$$

$$= \sum_{\theta,\iota,\kappa,\lambda,\alpha,\beta}\frac{\partial x_\theta}{\partial x'_\pi}\frac{\partial x_\kappa}{\partial x'_\sigma}\frac{\partial x'_\rho}{\partial x_\iota}\frac{\partial x'_\tau}{\partial x_\lambda}A(\alpha,\theta;\beta,\iota)B(\beta,\kappa;\alpha,\lambda).$$

Also by hypothesis, using (8),

$$A(\alpha,\theta;\beta,\iota) = \sum_{\mu,\gamma,\nu,\delta}\frac{\partial x'_\mu}{\partial x_\alpha}\frac{\partial x'_\gamma}{\partial x_\theta}\frac{\partial x_\beta}{\partial x'_\nu}\frac{\partial x_\iota}{\partial x'_\delta}A'(\mu,\gamma;\nu,\delta).$$

Hence

$$\sum_{\mu,\nu} A'(\mu,\pi;\nu,\rho)B'(\nu,\sigma;\mu,\tau)$$

$$= \sum_{\substack{\theta,\iota,\kappa,\lambda,\alpha,\beta \\ \mu,\gamma,\nu,\delta}}\frac{\partial x_\theta}{\partial x'_\pi}\frac{\partial x'_\gamma}{\partial x_\theta}\frac{\partial x_\iota}{\partial x'_\delta}\frac{\partial x'_\rho}{\partial x_\iota}\frac{\partial x_\beta}{\partial x'_\nu}\frac{\partial x_\kappa}{\partial x'_\sigma}\frac{\partial x'_\mu}{\partial x_\alpha}\frac{\partial x'_\tau}{\partial x_\lambda}A'(\mu,\gamma;\nu,\delta)B(\beta,\kappa;\alpha,\lambda)$$

$$= \sum_{\mu,\nu,\beta,\kappa,\alpha,\lambda}\frac{\partial x_\beta}{\partial x'_\nu}\frac{\partial x_\kappa}{\partial x'_\sigma}\frac{\partial x'_\mu}{\partial x_\alpha}\frac{\partial x'_\tau}{\partial x_\lambda}B(\beta,\kappa;\alpha,\lambda)\sum_{\theta,\iota,\gamma,\delta}\frac{\partial x_\theta}{\partial x'_\pi}\frac{\partial x'_\gamma}{\partial x_\theta}\frac{\partial x_\iota}{\partial x'_\delta}\frac{\partial x'_\rho}{\partial x_\iota}A'(\mu,\gamma;\nu,\delta).$$

By Lemma II, § 4,

$$\sum_{\theta,\iota,\gamma,\delta}\frac{\partial x_\theta}{\partial x'_\pi}\frac{\partial x'_\gamma}{\partial x_\theta}\frac{\partial x_\iota}{\partial x'_\delta}\frac{\partial x'_\rho}{\partial x_\iota}A'(\mu,\gamma;\nu,\delta) = A'(\mu,\pi;\nu,\rho).$$

Hence

$$\sum_{\mu,\nu} A'(\mu,\pi;\nu,\rho)B'(\nu,\sigma;\mu,\tau) = \sum_{\mu,\nu,\beta,\kappa,\alpha,\lambda} \frac{\partial x_\beta}{\partial x_\nu'}\frac{\partial x_\kappa}{\partial x_\sigma'}\frac{\partial x_\mu'}{\partial x_\alpha}\frac{\partial x_\tau'}{\partial x_\lambda} A'(\mu,\pi;\nu,\rho)B(\beta,\kappa;\alpha,\lambda),$$

or

$$\sum_{\mu,\nu} A'(\mu,\pi;\nu,\rho)\left[B'(\nu,\sigma;\mu,\tau) - \sum_{\beta,\kappa,\alpha,\lambda} \frac{\partial x_\beta}{\partial x_\nu'}\frac{\partial x_\kappa}{\partial x_\sigma'}\frac{\partial x_\mu'}{\partial x_\alpha}\frac{\partial x_\tau'}{\partial x_\lambda} B(\beta,\kappa;\alpha,\lambda)\right] = 0 .$$

Since by hypothesis this is true for every A, and by § 9 A is general and therefore A' general, we may conclude that the quantity in the bracket is zero, i.e.

$$B'(\nu,\sigma;\mu,\tau) = \sum_{\beta,\kappa,\alpha,\lambda} \frac{\partial x_\beta}{\partial x_\nu'}\frac{\partial x_\kappa}{\partial x_\sigma'}\frac{\partial x_\mu'}{\partial x_\alpha}\frac{\partial x_\tau'}{\partial x_\lambda} B(\beta,\kappa;\alpha,\lambda),$$

which is the condition that B is a tensor $B\{^{m+t}_{n+s}\}(\nu,\sigma;\mu,\tau)$.

§11. *Association of tensors.*

Let $g_{(2)}$ or $g_{\mu\nu}$ represent a symmetric covariant tensor of valence two with det $g \neq 0$; then by §8 there exists a tensor $g^{(2)}$ which is the reciprocal of $g_{\mu\nu}$.

Consider a tensor $A\{^{n}_{m}\}$. Let us form the outer product $g_{(2)}A\{^{n}_{m}\}$ and contract with respect to one index of $g_{(2)}$; since $g_{(2)}$ is symmetric it is immaterial which of its indices is used. The result of this operation is a tensor $A\{^{n-1}_{m+1}\}$; there are clearly n such tensors (which are distinct unless $A\{^{n}_{m}\}$ is symmetric in a pair of the n indices of contravariance). This process may be performed successively n times. Instead of employing $g_{(2)}$ we may employ $g^{(2)}$, and perform the operation up to m times. In this way we obtain a set of tensors $A\{^{m+n-k}_{k}\}$ which we shall call *associated tensors*. Given any two associated tensors $A\{^{m+n-k}_{k}\}$, $A\{^{m+n-k-l}_{k+l}\}$ we can obtain the second from the first by successively multiplying by $g_{(2)}$ or $g^{(2)}$ and contracting with respect to the proper indices.

In § 6 we saw that by equating a tensor to zero a covariant equation is obtained. We shall show that the equation

$$A^{(n)}_{(m)} = 0$$

is equivalent to

$$A^{(m+n-k)}_{(k)} = 0,$$

that is, that the same law is obtained by equating to zero any one of the associated tensors. In this fact lies the chief significance of association of tensors.

To prove this proposition it is sufficient to show that $A^{(n)}_{(m)}=0$ is equivalent to $A^{(n-1)}_{(m+1)}=0$. For simplicity we shall consider a tensor $A^{(2)}$; the extension to the general case is obvious. If $A^{\mu\nu}=0$ then obviously $\sum_\nu g_{\sigma\nu}A^{\mu\nu}=0$. Conversely, the second implies the first; for the determinant of the last equations in the $A^{\mu\nu}$ is $(\det g)^4$, which is different from zero.

CHAPTER III

DIFFERENTIAL PROPERTIES OF TENSORS

§12. *The Christoffel symbols.*

Before treating the differential properties of tensors it is convenient to introduce two symbols commonly used in Differential Geometry; these are known as the *Christoffel* symbols:

$$\begin{bmatrix}\mu\nu\\ \sigma\end{bmatrix} \equiv \frac{1}{2}\left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \frac{\partial g_{\nu\sigma}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\sigma}\right), \tag{16}$$

$$\begin{Bmatrix}\mu\nu\\ \sigma\end{Bmatrix} \equiv \sum_\alpha g^{\sigma\alpha}\begin{bmatrix}\mu\nu\\ \alpha\end{bmatrix}. \tag{17}$$

It will be readily seen by (23) that these expressions are not tensors. They have several properties, which we proceed to write down. Clearly each is symmetric in μ, ν. From (16)

$$\begin{bmatrix}\mu\nu\\ \nu\end{bmatrix} = \frac{1}{2}\,\frac{\partial g_{\nu\nu}}{\partial x_\mu}.$$

From (17)

$$\sum_\nu \begin{Bmatrix}\mu\nu\\ \nu\end{Bmatrix} = \frac{1}{2}\sum_{\nu,\alpha} g^{\nu\alpha}\left(\frac{\partial g_{\mu\alpha}}{\partial x_\nu} + \frac{\partial g_{\nu\alpha}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\alpha}\right).$$

Due to the symmetry of $g^{\nu\alpha}$

$$\sum_{\nu,\alpha} g^{\nu\alpha}\frac{\partial g_{\mu\alpha}}{\partial x_\nu} = \sum_{\nu,\alpha} g^{\nu\alpha}\frac{\partial g_{\mu\nu}}{\partial x_\alpha};$$

hence

$$\sum_\nu \begin{Bmatrix}\mu\nu\\ \nu\end{Bmatrix} = \frac{1}{2}\sum_{\nu,\alpha} g^{\nu\alpha}\frac{\partial g_{\nu\alpha}}{\partial x_\mu}. \tag{18}$$

From (16)

$$\begin{bmatrix}\mu\nu\\ \sigma\end{bmatrix} + \begin{bmatrix}\mu\sigma\\ \nu\end{bmatrix} = \frac{\partial g_{\nu\sigma}}{\partial x_\mu}. \tag{19}$$

From (17)

$$\sum_\sigma g_{\beta\sigma}\begin{Bmatrix}\mu\nu\\ \sigma\end{Bmatrix} = \sum_{\sigma,\alpha} g_{\beta\sigma}g^{\sigma\alpha}\begin{bmatrix}\mu\nu\\ \alpha\end{bmatrix} = \sum_\alpha I(\beta,\alpha)\begin{bmatrix}\mu\nu\\ \alpha\end{bmatrix} = \begin{bmatrix}\mu\nu\\ \beta\end{bmatrix}.$$

By a change of indices this becomes

$$(20)\qquad \begin{bmatrix}\mu\nu\\ \sigma\end{bmatrix} = \sum_{\alpha} g_{\sigma\alpha}\begin{Bmatrix}\mu\nu\\ \alpha\end{Bmatrix}.$$

From (19) it follows that $\left[{\mu\nu \atop \sigma}\right]=0$ implies $g_{\mu\nu}=\text{constant}$. From (20) it follows that $\left\{{\mu\nu \atop \sigma}\right\}=0$ implies $g_{\mu\nu}=\text{constant}$.

Now consider two tensors $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$, and let $\left[{\mu\nu \atop \sigma}\right]_{\bar{g}}$ be defined by (16) with $\bar{g}_{\mu\nu}$ in place of $g_{\mu\nu}$. Then

$$(21)\qquad \begin{bmatrix}\mu\nu\\ \sigma\end{bmatrix} - \begin{bmatrix}\mu\nu\\ \sigma\end{bmatrix}_{\bar{g}} = \begin{bmatrix}\mu\nu\\ \sigma\end{bmatrix}_{g-\bar{g}}.$$

Consequently $\left[{\mu\nu \atop \sigma}\right]=\left[{\mu\nu \atop \sigma}\right]_{\bar{g}}$ implies $g_{\mu\nu}=\bar{g}_{\mu\nu}+\text{constant}$.

§13. *Partial derivatives of the transformation.*

We have the relation

$$(22)\qquad g_{\mu\nu}' = \sum_{i,j} \frac{\partial x_i}{\partial x_\mu'}\frac{\partial x_j}{\partial x_\nu'} g_{ij}.$$

We wish to obtain the second derivatives of the x_i with respect to the x_i' in terms of the first derivatives of the x_i and the Christoffel symbols. Differentiating (22) we have

$$\frac{\partial g_{\mu\nu}'}{\partial x_\sigma'} = \sum_{i,j}\left[g_{ij}\left(\frac{\partial^2 x_i}{\partial x_\sigma'\partial x_\mu'}\frac{\partial x_j}{\partial x_\nu'} + \frac{\partial^2 x_j}{\partial x_\sigma'\partial x_\nu'}\frac{\partial x_i}{\partial x_\mu'}\right) + \frac{\partial x_i}{\partial x_\mu'}\frac{\partial x_j}{\partial x_\nu'}\frac{\partial g_{ij}}{\partial x_\sigma'}\right].$$

Since $g_{ij}=g_{ji}$ we may interchange i and j in the second term of the parenthesis. Noting also that

$$\frac{\partial g_{ij}}{\partial x_\sigma'} = \sum_k \frac{\partial g_{ij}}{\partial x_k}\frac{\partial x_k}{\partial x_\sigma'}$$

the above equation may be written

$$\frac{\partial g_{\mu\nu}'}{\partial x_\sigma'} = \sum_{i,j,k}\left[g_{ij}\left(\frac{\partial^2 x_i}{\partial x_\sigma'\partial x_\mu'}\frac{\partial x_j}{\partial x_\nu'} + \frac{\partial^2 x_i}{\partial x_\sigma'\partial x_\nu'}\frac{\partial x_j}{\partial x_\mu'}\right) + \frac{\partial x_i}{\partial x_\mu'}\frac{\partial x_j}{\partial x_\nu'}\frac{\partial x_k}{\partial x_\sigma'}\frac{\partial g_{ij}}{\partial x_k}\right].$$

Similarly

$$\frac{\partial g_{\nu\sigma}'}{\partial x_\mu'} = \sum_{i,j,k}\left[g_{ij}\left(\frac{\partial^2 x_i}{\partial x_\mu'\partial x_\nu'}\frac{\partial x_j}{\partial x_\sigma'} + \frac{\partial^2 x_i}{\partial x_\mu'\partial x_\sigma'}\frac{\partial x_j}{\partial x_\nu'}\right) + \frac{\partial x_j}{\partial x_\nu'}\frac{\partial x_k}{\partial x_\sigma'}\frac{\partial x_i}{\partial x_\mu'}\frac{\partial g_{jk}}{\partial x_i}\right],$$

$$\frac{\partial g_{\mu\sigma}'}{\partial x_\nu'} = \sum_{i,j,k}\left[g_{ij}\left(\frac{\partial^2 x_i}{\partial x_\nu'\partial x_\mu'}\frac{\partial x_j}{\partial x_\sigma'} + \frac{\partial^2 x_i}{\partial x_\nu'\partial x_\sigma'}\frac{\partial x_j}{\partial x_\mu'}\right) + \frac{\partial x_i}{\partial x_\mu'}\frac{\partial x_k}{\partial x_\sigma'}\frac{\partial x_j}{\partial x_\nu'}\frac{\partial g_{ki}}{\partial x_j}\right],$$

where in the last terms we have changed the order of the summation indices i, j, k.

If we add the last two and then subtract the first we obtain, using (16), and noting that

$$\frac{\partial^2 x_i}{\partial x_\mu' \partial x_\nu'} = \frac{\partial^2 x_i}{\partial x_\nu' \partial x_\mu'},$$

$$(23) \qquad \begin{bmatrix} \mu\nu \\ \sigma \end{bmatrix}' = \sum_{i,j} g_{ij} \frac{\partial^2 x_i}{\partial x_\mu' \partial x_\nu'} \frac{\partial x_j}{\partial x_\sigma'} + \sum_{i,j,k} \frac{\partial x_i}{\partial x_\mu'} \frac{\partial x_j}{\partial x_\nu'} \frac{\partial x_k}{\partial x_\sigma'} \begin{bmatrix} ij \\ k \end{bmatrix}.$$

Hence

$$\sum_{i,j} g_{ij} \frac{\partial^2 x_i}{\partial x_\mu' \partial x_\nu'} \frac{\partial x_j}{\partial x_\sigma'} = \begin{bmatrix} \mu\nu \\ \sigma \end{bmatrix}' - \sum_{i,j,k} \frac{\partial x_i}{\partial x_\mu'} \frac{\partial x_j}{\partial x_\nu'} \frac{\partial x_k}{\partial x_\sigma'} \begin{bmatrix} ij \\ k \end{bmatrix}.$$

Multiplying by $g'^{\sigma\rho} \dfrac{\partial x_e}{\partial x_\rho'}$ and summing as to ρ and σ this becomes

$$(24) \quad \sum_{i,j,\rho,\sigma} g_{ij} \frac{\partial x_e}{\partial x_\rho'} \frac{\partial x_j}{\partial x_\sigma'} g'^{\sigma\rho} \frac{\partial^2 x_i}{\partial x_\mu' \partial x_\nu'} = \sum_{\rho,\sigma} g'^{\sigma\rho} \frac{\partial x_e}{\partial x_\rho'} \begin{bmatrix} \mu\nu \\ \sigma \end{bmatrix}'$$
$$- \sum_{i,j,k,\rho,\sigma} \frac{\partial x_i}{\partial x_\mu'} \frac{\partial x_j}{\partial x_\nu'} \begin{bmatrix} ij \\ k \end{bmatrix} \frac{\partial x_e}{\partial x_\rho'} \frac{\partial x_k}{\partial x_\sigma'} g'^{\sigma\rho}.$$

Since g^{ij} is a contravariant tensor the first member becomes

$$\sum_{i,j} g_{ij} g^{je} \frac{\partial^2 x_i}{\partial x_\mu' \partial x_\nu'} = \frac{\partial^2 x_e}{\partial x_\mu' \partial x_\nu'}.$$

The first term of the second member of (24) becomes, by (17)

$$\sum_\rho \frac{\partial x_e}{\partial x_\rho'} \begin{Bmatrix} \mu\nu \\ \rho \end{Bmatrix}'.$$

The second term becomes

$$-\sum_{i,j,k} \frac{\partial x_i}{\partial x_\mu'} \frac{\partial x_j}{\partial x_\nu'} g^{ek} \begin{bmatrix} ij \\ k \end{bmatrix} = -\sum_{i,j} \frac{\partial x_i}{\partial x_\mu'} \frac{\partial x_j}{\partial x_\nu'} \begin{Bmatrix} ij \\ e \end{Bmatrix}.$$

Hence (24) may be written

$$(25) \qquad \frac{\partial^2 x_e}{\partial x_\mu' \partial x_\nu'} = \sum_\rho \frac{\partial x_e}{\partial x_\rho'} \begin{Bmatrix} \mu\nu \\ \rho \end{Bmatrix}' - \sum_{i,j} \frac{\partial x_i}{\partial x_\mu'} \frac{\partial x_j}{\partial x_\nu'} \begin{Bmatrix} ij \\ e \end{Bmatrix},$$

which is the desired result.

To find an expression for $\partial^2 x_e' / \partial x_\mu \partial x_\nu$ we proceed similarly, starting with the expression

$$g_{\mu\nu} = \sum_{i,j} \frac{\partial x_i'}{\partial x_\mu} \frac{\partial x_j'}{\partial x_\nu} g_{ij}'.$$

We obtain

$$\begin{bmatrix}\mu\nu\\ \sigma\end{bmatrix} = \sum_{i,j} g'_{ij}\frac{\partial^2 x'_i}{\partial x_\mu \partial x_\nu}\frac{\partial x'_j}{\partial x_\sigma} + \sum_{i,j,k}\frac{\partial x'_i}{\partial x_\mu}\frac{\partial x'_j}{\partial x_\nu}\frac{\partial x'_k}{\partial x_\sigma}\begin{bmatrix}i\,j\\ k\end{bmatrix}'.$$

We multiply by $g^{\sigma\rho}\partial x'_e/\partial x_\rho$, sum as to ρ and σ, and proceed as before, obtaining

$$\frac{\partial^2 x'_e}{\partial x_\mu \partial x_\nu} = \sum_\rho \frac{\partial x'_e}{\partial x_\rho}\begin{Bmatrix}\mu\nu\\ \rho\end{Bmatrix} - \sum_{i,j}\frac{\partial x'_i}{\partial x_\mu}\frac{\partial x'_j}{\partial x_\nu}\begin{Bmatrix}i\,j\\ e\end{Bmatrix}'. \tag{26}$$

§14. *The mixed Riemann-Christoffel tensor.*

By (25)

$$\frac{\partial^2 x_e}{\partial x'_\nu \partial x'_\mu} = -\sum_{\alpha,j}\frac{\partial x_\alpha}{\partial x'_\mu}\frac{\partial x_j}{\partial x'_\nu}\begin{Bmatrix}\alpha j\\ e\end{Bmatrix} + \sum_\rho \frac{\partial x_e}{\partial x'_\rho}\begin{Bmatrix}\mu\,\nu\\ \rho\end{Bmatrix}'$$

and

$$\frac{\partial^2 x_e}{\partial x'_\lambda \partial x'_\mu} = -\sum_{\alpha,j}\frac{\partial x_\alpha}{\partial x'_\mu}\frac{\partial x_j}{\partial x'_\lambda}\begin{Bmatrix}\alpha j\\ e\end{Bmatrix} + \sum_\rho \frac{\partial x_e}{\partial x'_\rho}\begin{Bmatrix}\mu\,\lambda\\ \rho\end{Bmatrix}'.$$

From the first of these differentiated with respect to x'_λ subtract the second differentiated with respect to x'_ν:

$$\begin{aligned}\frac{\partial^3 x_e}{\partial x'_\lambda \partial x'_\nu \partial x'_\mu} - \frac{\partial^3 x_e}{\partial x'_\nu \partial x'_\lambda \partial x'_\mu} &= \sum_{\alpha,j}\frac{\partial^2 x_\alpha}{\partial x'_\nu \partial x'_\mu}\frac{\partial x_j}{\partial x'_\lambda}\begin{Bmatrix}\alpha j\\ e\end{Bmatrix}\\
&+ \sum_{\alpha,j}\frac{\partial x_\alpha}{\partial x'_\mu}\frac{\partial^2 x_j}{\partial x'_\nu \partial x'_\lambda}\begin{Bmatrix}\alpha j\\ e\end{Bmatrix} + \sum_{\alpha,j}\frac{\partial x_\alpha}{\partial x'_\mu}\frac{\partial x_j}{\partial x'_\lambda}\frac{\partial}{\partial x'_\nu}\begin{Bmatrix}\alpha j\\ e\end{Bmatrix}\\
&- \sum_{\alpha,j}\frac{\partial^2 x_\alpha}{\partial x'_\lambda \partial x'_\mu}\frac{\partial x_j}{\partial x'_\nu}\begin{Bmatrix}\alpha j\\ e\end{Bmatrix} - \sum_{\alpha,j}\frac{\partial x_\alpha}{\partial x'_\mu}\frac{\partial^2 x_j}{\partial x'_\lambda \partial x'_\nu}\begin{Bmatrix}\alpha j\\ e\end{Bmatrix}\\
&- \sum_{\alpha,j}\frac{\partial x_\alpha}{\partial x'_\mu}\frac{\partial x_j}{\partial x'_\nu}\frac{\partial}{\partial x'_\lambda}\begin{Bmatrix}\alpha j\\ e\end{Bmatrix} + \sum_\rho \frac{\partial^2 x_e}{\partial x'_\lambda \partial x'_\rho}\begin{Bmatrix}\mu\,\nu\\ \rho\end{Bmatrix}\\
&+ \sum_\rho \frac{\partial x_e}{\partial x'_\rho}\frac{\partial}{\partial x'_\lambda}\begin{Bmatrix}\mu\,\nu\\ \rho\end{Bmatrix}' - \sum_\rho \frac{\partial^2 x_e}{\partial x'_\nu \partial x'_\rho}\begin{Bmatrix}\mu\,\lambda\\ \rho\end{Bmatrix}'\\
&- \sum_\rho \frac{\partial x_e}{\partial x'_\rho}\frac{\partial}{\partial x'_\nu}\begin{Bmatrix}\mu\,\lambda\\ \rho\end{Bmatrix}'.\end{aligned} \tag{27}$$

The second and fifth terms of the right member cancel each other. The third term may be written

$$\sum_{\alpha,j,k} \frac{\partial x_\alpha}{\partial x'_\mu} \frac{\partial x_j}{\partial x'_\lambda} \frac{\partial x_k}{\partial x'_\nu} \frac{\partial}{\partial x_k} \begin{Bmatrix} \alpha j \\ e \end{Bmatrix},$$

and the sixth may be written

$$= - \sum_{\alpha,j,k} \frac{\partial x_\alpha}{\partial x'_\mu} \frac{\partial x_j}{\partial x'_\nu} \frac{\partial x_k}{\partial x'_\lambda} \frac{\partial}{\partial x_k} \begin{Bmatrix} \alpha j \\ e \end{Bmatrix}.$$

Hence the right member may be written

$$\sum_{\alpha,j,k} \frac{\partial x_\alpha}{\partial x'_\mu} \frac{\partial x_j}{\partial x'_\nu} \frac{\partial x_k}{\partial x'_\lambda} \left(\frac{\partial}{\partial x_j} \begin{Bmatrix} \alpha k \\ e \end{Bmatrix} - \frac{\partial}{\partial x_k} \begin{Bmatrix} \alpha j \\ e \end{Bmatrix} \right)$$

$$+ \sum_{m,k} \begin{Bmatrix} m k \\ e \end{Bmatrix} \left(\frac{\partial^2 x_m}{\partial x'_\nu \partial x'_\mu} \frac{\partial x_k}{\partial x'_\lambda} - \frac{\partial^2 x_m}{\partial x'_\lambda \partial x'_\mu} \frac{\partial x_k}{\partial x'_\nu} \right)$$

$$+ \sum_{\rho} \frac{\partial x_e}{\partial x'_\rho} \left(\frac{\partial}{\partial x'_\lambda} \begin{Bmatrix} \mu \nu \\ \rho \end{Bmatrix}' - \frac{\partial}{\partial x'_\nu} \begin{Bmatrix} \mu \lambda \\ \rho \end{Bmatrix}' \right)$$

$$+ \sum_{\rho} \frac{\partial^2 x_e}{\partial x'_\lambda \partial x'_\rho} \begin{Bmatrix} \mu \nu \\ \rho \end{Bmatrix}' - \sum_{\rho} \frac{\partial^2 x_e}{\partial x'_\nu \partial x'_\rho} \begin{Bmatrix} \mu \lambda \\ \rho \end{Bmatrix}'.$$

Now by (25) the second term may be written

$$\sum_{m,k} \begin{Bmatrix} m k \\ e \end{Bmatrix} \Bigg[\sum_{\beta} \frac{\partial x_m}{\partial x'_\beta} \begin{Bmatrix} \mu \nu \\ \beta \end{Bmatrix}' \frac{\partial x_k}{\partial x'_\lambda} - \sum_{i,j} \frac{\partial x_i}{\partial x'_\mu} \frac{\partial x_j}{\partial x'_\nu} \begin{Bmatrix} i j \\ m \end{Bmatrix} \frac{\partial x_k}{\partial x'_\lambda}$$

$$- \sum_{\beta} \frac{\partial x_m}{\partial x'_\beta} \begin{Bmatrix} \mu \lambda \\ \beta \end{Bmatrix}' \frac{\partial x_k}{\partial x'_\nu} + \sum_{i,j} \frac{\partial x_i}{\partial x'_\mu} \frac{\partial x_j}{\partial x'_\lambda} \begin{Bmatrix} i j \\ m \end{Bmatrix} \frac{\partial x_k}{\partial x'_\nu} \Bigg],$$

and the last two terms may be written

$$\sum_{\rho} \begin{Bmatrix} \mu \nu \\ \rho \end{Bmatrix}' \left(\sum_{\beta} \begin{Bmatrix} \rho \lambda \\ \beta \end{Bmatrix}' \frac{\partial x_e}{\partial x'_\beta} - \sum_{m,k} \frac{\partial x_m}{\partial x'_\rho} \frac{\partial x_k}{\partial x'_\lambda} \begin{Bmatrix} m k \\ e \end{Bmatrix} \right)$$

$$- \sum_{\rho} \begin{Bmatrix} \mu \lambda \\ \rho \end{Bmatrix}' \left(\sum_{\beta} \begin{Bmatrix} \rho \nu \\ \beta \end{Bmatrix}' \frac{\partial x_e}{\partial x'_\beta} - \sum_{m,k} \frac{\partial x_m}{\partial x'_\rho} \frac{\partial x_k}{\partial x'_\nu} \begin{Bmatrix} m k \\ e \end{Bmatrix} \right).$$

Hence the right member of (27) is

$$\sum_{i,j,k} \frac{\partial x_i}{\partial x'_\mu} \frac{\partial x_j}{\partial x'_\nu} \frac{\partial x_k}{\partial x'_\lambda}\left(\frac{\partial}{\partial x_j}\begin{Bmatrix} ik \\ e \end{Bmatrix} - \frac{\partial}{\partial x_k}\begin{Bmatrix} ij \\ e \end{Bmatrix}\right)$$

$$- \sum_{i,j,k,m} \frac{\partial x_i}{\partial x'_\mu} \frac{\partial x_j}{\partial x'_\nu} \frac{\partial x_k}{\partial x'_\lambda}\begin{Bmatrix} m\,k \\ e \end{Bmatrix}\begin{Bmatrix} ij \\ m \end{Bmatrix}$$

$$+ \sum_{i,j,m,k} \frac{\partial x_i}{\partial x'_\mu} \frac{\partial x_j}{\partial x'_\nu} \frac{\partial x_k}{\partial x'_\lambda}\begin{Bmatrix} m\,j \\ e \end{Bmatrix}\begin{Bmatrix} ik \\ m \end{Bmatrix} + \sum_{\beta,m,k} \frac{\partial x_m}{\partial x'_\beta} \frac{\partial x_k}{\partial x'_\lambda}\begin{Bmatrix} m\,k \\ e \end{Bmatrix}\begin{Bmatrix} \mu\,\nu \\ \beta \end{Bmatrix}'$$

$$- \sum_{\beta,m,k} \frac{\partial x_m}{\partial x'_\beta} \frac{\partial x_k}{\partial x'_\nu}\begin{Bmatrix} m\,k \\ e \end{Bmatrix}\begin{Bmatrix} \mu\,\lambda \\ \beta \end{Bmatrix}' + \sum_{\rho} \frac{\partial x_e}{\partial x'_\rho}\left(\frac{\partial}{\partial x'_\lambda}\begin{Bmatrix} \mu\,\nu \\ \rho \end{Bmatrix}' - \frac{\partial}{\partial x'_\nu}\begin{Bmatrix} \mu\,\lambda \\ \rho \end{Bmatrix}'\right)$$

$$+ \sum_{\rho,\beta} \frac{\partial x_e}{\partial x'_\beta}\begin{Bmatrix} \mu\,\nu \\ \rho \end{Bmatrix}'\begin{Bmatrix} \rho\,\lambda \\ \beta \end{Bmatrix}' - \sum_{\rho,\beta} \frac{\partial x_e}{\partial x'_\beta}\begin{Bmatrix} \mu\,\lambda \\ \rho \end{Bmatrix}'\begin{Bmatrix} \rho\,\nu \\ \beta \end{Bmatrix}'$$

$$- \sum_{\rho,m,k} \frac{\partial x_m}{\partial x'_\rho} \frac{\partial x_k}{\partial x'_\lambda}\begin{Bmatrix} \mu\,\nu \\ \rho \end{Bmatrix}'\begin{Bmatrix} m\,k \\ e \end{Bmatrix} + \sum_{\rho,m,k} \frac{\partial x_m}{\partial x'_\rho} \frac{\partial x_k}{\partial x'_\nu}\begin{Bmatrix} \mu\,\lambda \\ \rho \end{Bmatrix}'\begin{Bmatrix} m\,k \\ e \end{Bmatrix}.$$

The fourth and ninth terms cancel each other, and the fifth and tenth likewise. Hence the right member of (27) is

$$\sum_{i,j,k} \frac{\partial x_i}{\partial x'_\mu} \frac{\partial x_j}{\partial x'_\nu} \frac{\partial x_k}{\partial x'_\lambda}\left[\frac{\partial}{\partial x_j}\begin{Bmatrix} ik \\ e \end{Bmatrix} - \frac{\partial}{\partial x_k}\begin{Bmatrix} ij \\ e \end{Bmatrix} + \sum_m \begin{Bmatrix} ik \\ m \end{Bmatrix}\begin{Bmatrix} mj \\ e \end{Bmatrix}\right.$$

$$\left. - \sum_m \begin{Bmatrix} ij \\ m \end{Bmatrix}\begin{Bmatrix} m\,k \\ e \end{Bmatrix}\right] + \sum_\rho \frac{\partial x_e}{\partial x'_\rho}\left[\frac{\partial}{\partial x'_\lambda}\begin{Bmatrix} \mu\nu \\ \rho \end{Bmatrix}' - \frac{\partial}{\partial x'_\nu}\begin{Bmatrix} \mu\lambda \\ \rho \end{Bmatrix}'\right.$$

$$\left. + \sum_\beta \begin{Bmatrix} \mu\nu \\ \beta \end{Bmatrix}'\begin{Bmatrix} \beta\lambda \\ \rho \end{Bmatrix}' - \sum_\beta \begin{Bmatrix} \mu\lambda \\ \beta \end{Bmatrix}'\begin{Bmatrix} \beta\nu \\ \rho \end{Bmatrix}'\right].$$

Let us now define

$$(28) \qquad R(\mu,\nu,\lambda,\rho) \equiv \sum_\beta \begin{Bmatrix} \mu\lambda \\ \beta \end{Bmatrix}\begin{Bmatrix} \beta\nu \\ \rho \end{Bmatrix} - \sum_\beta \begin{Bmatrix} \mu\nu \\ \beta \end{Bmatrix}\begin{Bmatrix} \beta\lambda \\ \rho \end{Bmatrix} + \frac{\partial}{\partial x_\nu}\begin{Bmatrix} \mu\lambda \\ \rho \end{Bmatrix} - \frac{\partial}{\partial x_\lambda}\begin{Bmatrix} \mu\nu \\ \rho \end{Bmatrix}.$$

Then (27) becomes

$$(29)\qquad \frac{\partial^3 x_e}{\partial x'_\lambda \partial x'_\nu \partial x'_\mu} - \frac{\partial^3 x_e}{\partial x'_\nu \partial x'_\lambda \partial x'_\mu} = -\sum_\rho \frac{\partial x_e}{\partial x'_\rho}\Big[R'(\mu,\nu,\lambda,\rho)$$
$$- \sum_{i,j,k,l} \frac{\partial x_i}{\partial x'_\mu}\,\frac{\partial x_j}{\partial x'_\nu}\,\frac{\partial x_k}{\partial x'_\lambda}\,\frac{\partial x'_\rho}{\partial x_l}R(i,j,k,l)\Big].$$

Now, since the left member of (29) is zero, it follows that

$$R'(\mu,\nu,\lambda,\rho) = \sum_{i,j,k,l} \frac{\partial x_i}{\partial x'_\mu}\,\frac{\partial x_j}{\partial x'_\nu}\,\frac{\partial x_k}{\partial x'_\lambda}\,\frac{\partial x'_\rho}{\partial x_l}R(i,j,k,l).$$

Hence $R(\mu, \nu, \lambda, \rho)$ is a tensor $R^\rho_{\mu\nu\lambda}$; this is called the *mixed Riemann-Christoffel tensor*, and is of fundamental significance in the Theory of Relativity.† This tensor is evidently antisymmetric in ν and λ.

§15. *Covariant differentiation of tensors.*

We now come to the consideration of differential properties of tensors. Let $A_\mu(h_\nu)$ be a covariant tensor of valence one. Then

$$(30)\qquad A'_\mu = \sum_\alpha \frac{\partial x_\alpha}{\partial x'_\mu}A_\alpha.$$

Hence

$$(31)\qquad \frac{\partial A'_\mu}{\partial x'_\sigma} = \sum_\alpha \frac{\partial^2 x_\alpha}{\partial x'_\mu \partial x'_\sigma}A_\alpha + \sum_\alpha \frac{\partial x_\alpha}{\partial x'_\mu}\,\frac{\partial A_\alpha}{\partial x'_\sigma}.$$

Substituting in the first term of the right member according to (25) this term becomes

$$\sum_{\alpha,\rho}\left\{\begin{matrix}\mu\,\sigma\\ \rho\end{matrix}\right\}'\frac{\partial x_\alpha}{\partial x_\rho}A_\alpha - \sum_{\alpha,i,j,} \frac{\partial x_i}{\partial x'_\mu}\,\frac{\partial x_j}{\partial x'_\sigma}\left\{\begin{matrix}i\,j\\ \alpha\end{matrix}\right\}A_\alpha .$$

By (30) the first term of this last expression is

$$\sum_\rho \left\{\begin{matrix}\mu\,\sigma\\ \rho\end{matrix}\right\}' A'_\rho = \sum_\alpha \left\{\begin{matrix}\mu\,\sigma\\ \alpha\end{matrix}\right\}' A'_\alpha .$$

The second term of the right member of (31) may be written

$$\sum_i \frac{\partial x_i}{\partial x'_\mu}\,\frac{\partial A_i}{\partial x'_\sigma} = \sum_{i,j} \frac{\partial x_i}{\partial x'_\mu}\,\frac{\partial x_j}{\partial x'_\sigma}\,\frac{\partial A_i}{\partial x_j}.$$

† For the method of this section see Juvet: *Calcul Tensoriel*, 1922, p. 63.

Therefore (31) becomes

$$\frac{\partial A'_\mu}{\partial x'_\sigma} = \sum_\alpha \left\{ {\mu\sigma \atop \alpha} \right\}' A'_\alpha - \sum_{i,j} \frac{\partial x_i}{\partial x'_\mu} \frac{\partial x_j}{\partial x'_\sigma} \sum_\alpha \left\{ {ij \atop \alpha} \right\} A_\alpha + \sum_{i,j} \frac{\partial x_i}{\partial x'_\mu} \frac{\partial x_j}{\partial x'_\sigma} \frac{\partial A_i}{\partial x_j}.$$

Hence

$$\frac{\partial A'_\mu}{\partial x'_\sigma} - \sum_\alpha \left\{ {\mu\sigma \atop \alpha} \right\}' A'_\alpha = \sum_{i,j} \frac{\partial x_i}{\partial x'_\mu} \frac{\partial x_j}{\partial x'_\sigma} \left[\frac{\partial A_i}{\partial x_j} - \sum_\alpha \left\{ {ij \atop \alpha} \right\} A_\alpha \right].$$

By the last equation of §2 this shows that

$$(32) \qquad \frac{\mathfrak{D} A_\mu}{\mathfrak{D} x_\sigma} \equiv \frac{\partial A_\mu}{\partial x_\sigma} - \sum_\alpha \left\{ {\mu\sigma \atop \alpha} \right\} A_\alpha$$

is a covariant tensor of valence two.

We note in passing that $\partial A_\mu / \partial x_\sigma$ is not a tensor; for if it were then by (32) it would follow that $\sum_\alpha \left\{ {\mu\sigma \atop \alpha} \right\} A_\alpha$ is a tensor, and then by §7 that $\left\{ {\mu\sigma \atop \alpha} \right\}$ is a tensor, which, as stated in §12, is not the case.

Now let $A^\mu(h_i)$ be a contravariant tensor of valence one. Then

$$A'^\nu = \sum_\alpha \frac{\partial x'_\nu}{\partial x_\alpha} A^\alpha.$$

Hence

$$(33) \qquad \frac{\partial A'^\nu}{\partial x'_\sigma} = \sum_\alpha \frac{\partial^2 x'_\nu}{\partial x'_\sigma \partial x_\alpha} A^\alpha + \sum_\alpha \frac{\partial x'_\nu}{\partial x_\alpha} \frac{\partial A^\alpha}{\partial x'_\sigma}.$$

The first term of the right member may be written

$$\sum_{\alpha,\beta} \frac{\partial^2 x'_\nu}{\partial x_\beta \partial x_\alpha} \frac{\partial x_\beta}{\partial x'_\sigma} A^\alpha.$$

By (26) this equals

$$\sum_{\alpha,\beta,\rho} \left\{ {\alpha\beta \atop \rho} \right\} \frac{\partial x'_\nu}{\partial x_\rho} \frac{\partial x_\beta}{\partial x'_\sigma} A^\alpha - \sum_{\alpha,\beta,i,j} \frac{\partial x'_i}{\partial x_\alpha} \frac{\partial x'_j}{\partial x_\beta} \left\{ {ij \atop \nu} \right\}' \frac{\partial x_\beta}{\partial x'_\sigma} A^\alpha$$

$$= \sum_{j,i} \frac{\partial x_j}{\partial x'_\sigma} \frac{\partial x'_\nu}{\partial x_i} \sum_\alpha \left\{ {\alpha j \atop i} \right\} A^\alpha - \sum_{\alpha,i,j} \frac{\partial x'_i}{\partial x_\alpha} \left\{ {ij \atop \nu} \right\}' A^\alpha \sum_\beta \frac{\partial x'_j}{\partial x_\beta} \frac{\partial x_\beta}{\partial x'_\sigma}.$$

The last term of this expression is

$$- \sum_{\alpha,i} \frac{\partial x'_i}{\partial x_\alpha} \left\{ {i\sigma \atop \nu} \right\}' A^\alpha = - \sum_i \left\{ {i\sigma \atop \nu} \right\}' A'^i = - \sum_\alpha \left\{ {\alpha\sigma \atop \nu} \right\}' A'^\alpha.$$

The second term of the right member of (33) may be written

$$\sum_{\alpha,\beta} \frac{\partial x_\nu'}{\partial x_\alpha}\frac{\partial x_\beta}{\partial x_\sigma'}\frac{\partial A^\alpha}{\partial x_\beta} = \sum_{i,j} \frac{\partial x_\nu'}{\partial x_i}\frac{\partial x_j}{\partial x_\sigma'}\frac{\partial A^i}{\partial x_j}.$$

Hence (33) becomes

$$\frac{\partial A'^\nu}{\partial x_\sigma'} = \sum_{j,i} \frac{\partial x_j}{\partial x_\sigma'}\frac{\partial x_\nu'}{\partial x_i} \sum_\alpha \begin{Bmatrix} \alpha j \\ i \end{Bmatrix} A^\alpha - \sum_\alpha \begin{Bmatrix} \alpha\sigma \\ \nu \end{Bmatrix}' A'^\alpha + \sum_{i,j} \frac{\partial x_\nu'}{\partial x_i}\frac{\partial x_j}{\partial x_\sigma'}\frac{\partial A^i}{\partial x_j}.$$

Therefore

$$\frac{\partial A'^\nu}{\partial x_\sigma'} + \sum_\alpha \begin{Bmatrix} \alpha\sigma \\ \nu \end{Bmatrix}' A'^\alpha = \sum_{i,j} \frac{\partial x_\nu'}{\partial x_i}\frac{\partial x_j}{\partial x_\sigma'}\left[\frac{\partial A^i}{\partial x_j} + \sum_\alpha \begin{Bmatrix} \alpha j \\ i \end{Bmatrix} A^\alpha\right].$$

This shows that

$$\frac{\mathfrak{D}A^\nu}{\mathfrak{D}x_\sigma} \equiv \frac{\partial A^\nu}{\partial x_\sigma} + \sum_\alpha \begin{Bmatrix} \alpha\sigma \\ \nu \end{Bmatrix} A^\alpha$$

is a mixed tensor of type $\binom{1}{1}$, covariant in σ.

By direct extension of this process we find that for a tensor $A(h_\mu)$ of type $\binom{n}{m}$

$$(34)\qquad \frac{\mathfrak{D}A^{\nu_1,\ldots,\nu_n}_{\mu_1,\ldots,\mu_m}}{\mathfrak{D}x_\sigma} \equiv \frac{\partial A^{\nu_1,\ldots,\nu_n}_{\mu_1,\ldots,\mu_m}}{\partial x_\sigma} - \sum_{i=1}^{m}\sum_\alpha \begin{Bmatrix} \mu_i\sigma \\ \alpha \end{Bmatrix} A^{\nu_1,\ldots,\nu_n}_{\mu_1,\ldots,\mu_{i-1},\alpha,\mu_{i+1},\ldots,\mu_m} + \sum_{i=1}^{n}\sum_\alpha \begin{Bmatrix} \alpha\sigma \\ \nu_i \end{Bmatrix} A^{\nu_1,\ldots,\nu_{i-1},\alpha,\nu_{i+1},\ldots,\nu_n}_{\mu_1,\ldots,\mu_m}$$

is a tensor of type $\binom{n}{m+1}$, covariant in $\mu_1, \cdots, \mu_m, \sigma$.

If $m=n=0$ we shall define

$$\frac{\mathfrak{D}A^{(0)}_{(0)}}{\mathfrak{D}x_\sigma} \equiv \frac{\partial A^{(0)}_{(0)}}{\partial x_\sigma}.$$

§16. *Order of covariant differentiation.*

The question naturally arises whether the order of covariant differentiation is immaterial, that is whether

$$(35)\qquad \frac{\mathfrak{D}\left(\dfrac{\mathfrak{D}A_\mu}{\mathfrak{D}x_\nu}\right)}{\mathfrak{D}x_\lambda} - \frac{\mathfrak{D}\left(\dfrac{\mathfrak{D}A_\mu}{\mathfrak{D}x_\lambda}\right)}{\mathfrak{D}x_\nu}$$

is zero. We have, by (34) and (32),

$$\frac{\mathfrak{D}\left(\frac{\mathfrak{D}A_\mu}{\mathfrak{D}x_\nu}\right)}{\mathfrak{D}x_\lambda} = \frac{\partial}{\partial x_\lambda}\left(\frac{\mathfrak{D}A_\mu}{\mathfrak{D}x_\nu}\right) - \sum_\alpha \begin{Bmatrix}\mu\lambda\\ \alpha\end{Bmatrix}\frac{\mathfrak{D}A_\alpha}{\mathfrak{D}x_\nu} - \sum_\alpha \begin{Bmatrix}\nu\lambda\\ \alpha\end{Bmatrix}\frac{\mathfrak{D}A_\mu}{\mathfrak{D}x_\alpha}$$

$$= \frac{\partial}{\partial x_\lambda}\left[\frac{\partial A_\mu}{\partial x_\nu} - \sum_\beta \begin{Bmatrix}\mu\nu\\ \beta\end{Bmatrix}A_\beta\right] - \sum_\alpha \begin{Bmatrix}\mu\lambda\\ \alpha\end{Bmatrix}\left[\frac{\partial A_\alpha}{\partial x_\nu} - \sum_\beta \begin{Bmatrix}\alpha\nu\\ \beta\end{Bmatrix}A_\beta\right]$$

$$- \sum_\alpha \begin{Bmatrix}\nu\lambda\\ \alpha\end{Bmatrix}\left[\frac{\partial A_\mu}{\partial x_\alpha} - \sum_\beta \begin{Bmatrix}\mu\alpha\\ \beta\end{Bmatrix}A_\beta\right]$$

$$= \frac{\partial^2 A_\mu}{\partial x_\lambda \partial x_\nu} - \sum_\beta \begin{Bmatrix}\mu\nu\\ \beta\end{Bmatrix}\frac{\partial A_\beta}{\partial x_\lambda} - \sum_\beta A_\beta \frac{\partial}{\partial x_\lambda}\begin{Bmatrix}\mu\nu\\ \beta\end{Bmatrix}$$

$$- \sum_\alpha \begin{Bmatrix}\mu\lambda\\ \alpha\end{Bmatrix}\frac{\partial A_\alpha}{\partial x_\nu} + \sum_{\alpha,\beta} \begin{Bmatrix}\mu\lambda\\ \alpha\end{Bmatrix}\begin{Bmatrix}\alpha\nu\\ \beta\end{Bmatrix}A_\beta$$

$$- \sum_\alpha \begin{Bmatrix}\nu\lambda\\ \alpha\end{Bmatrix}\frac{\partial A_\mu}{\partial x_\alpha} + \sum_{\alpha,\beta} \begin{Bmatrix}\nu\lambda\\ \alpha\end{Bmatrix}\begin{Bmatrix}\mu\alpha\\ \beta\end{Bmatrix}A_\beta$$

$$= \frac{\partial^2 A_\mu}{\partial x_\lambda \partial x_\nu} - \sum_\beta \begin{Bmatrix}\mu\nu\\ \beta\end{Bmatrix}\frac{\partial A_\beta}{\partial x_\lambda} - \sum_\alpha \begin{Bmatrix}\mu\lambda\\ \alpha\end{Bmatrix}\frac{\partial A_\alpha}{\partial x_\nu}$$

$$- \sum_\alpha \begin{Bmatrix}\nu\lambda\\ \alpha\end{Bmatrix}\frac{\partial A_\mu}{\partial x_\alpha} + \sum_{\alpha,\beta} \begin{Bmatrix}\nu\lambda\\ \alpha\end{Bmatrix}\begin{Bmatrix}\mu\alpha\\ \beta\end{Bmatrix}A_\beta$$

$$+ \sum_\beta A_\beta\left[\sum_\alpha \begin{Bmatrix}\mu\lambda\\ \alpha\end{Bmatrix}\begin{Bmatrix}\alpha\nu\\ \beta\end{Bmatrix} - \frac{\partial}{\partial x_\lambda}\begin{Bmatrix}\mu\nu\\ \beta\end{Bmatrix}\right].$$

The sum of the first five terms of this right member is symmetric in ν and λ. Hence (35) has the value

$$\sum_\beta A_\beta\left[\sum_\alpha \begin{Bmatrix}\mu\lambda\\ \alpha\end{Bmatrix}\begin{Bmatrix}\alpha\nu\\ \beta\end{Bmatrix} - \sum_\alpha \begin{Bmatrix}\mu\nu\\ \alpha\end{Bmatrix}\begin{Bmatrix}\alpha\lambda\\ \beta\end{Bmatrix} - \frac{\partial}{\partial x_\lambda}\begin{Bmatrix}\mu\nu\\ \beta\end{Bmatrix} + \frac{\partial}{\partial x_\nu}\begin{Bmatrix}\mu\lambda\\ \beta\end{Bmatrix}\right],$$

or by (28)

$$(36) \qquad \sum_\beta A_\beta R(\mu,\nu,\lambda,\beta).$$

We see by (34) that (35) is the difference of two tensors of type $\binom{0}{3}$, and hence (36) is a tensor of type $\binom{0}{3}$. Hence by the quotient theorem of §7, $R(\mu, \nu, \lambda, \beta,)$ is a tensor $R^\beta_{\mu\nu\lambda}$, as was proved in §14.

Hence we have shown that a necessary and sufficient condition that the order of covariant differentiation of a tensor of type $\binom{0}{1}$ be immaterial is that the mixed Riemann-Christoffel tensor vanish.

§17. *Properties of covariant differentiation.*

Having defined a type of tensor differentiation we now consider some of its properties. Evidently

$$\frac{\mathfrak{D}(A \pm B)}{\mathfrak{D}x_\sigma} = \frac{\mathfrak{D}A}{\mathfrak{D}x_\sigma} \pm \frac{\mathfrak{D}B}{\mathfrak{D}x_\sigma}.$$

For every two tensors $A_{(m)}^{(n)}$, $B_{(r)}^{(t)}$ it is true that

$$\frac{\mathfrak{D}(AB)}{\mathfrak{D}x_\sigma} = A\frac{\mathfrak{D}B}{\mathfrak{D}x_\sigma} + B\frac{\mathfrak{D}A}{\mathfrak{D}x_\sigma}. \tag{37}$$

For

$$\begin{aligned}\frac{\mathfrak{D}(A_{\mu_1,\cdots,\mu_m}^{\nu_1,\cdots,\nu_n}B_{\rho_1,\cdots,\rho_r}^{\tau_1,\cdots,\tau_t})}{\mathfrak{D}x_\sigma} &= \frac{\partial A_\mu^\nu}{\partial x_\sigma}B_\rho^\tau + \frac{\partial B_\rho^\tau}{\partial x_\sigma}A_\mu^\nu \\ &- \sum_{i=1}^m \sum_\alpha \begin{Bmatrix}\mu_i\sigma\\ \alpha\end{Bmatrix} A_{\mu_1,\cdots,\mu_{i-1},\alpha,\mu_{i+1},\cdots,\mu_m}^\nu B_\rho^\tau \\ &- \sum_{i=1}^r \sum_\alpha \begin{Bmatrix}\rho_i\sigma\\ \alpha\end{Bmatrix} B_{\rho_1,\cdots,\rho_{i-1},\alpha,\rho_{i+1},\cdots,\rho_r}^\tau A_\mu^\nu \\ &+ \sum_{i=1}^n \sum_\alpha \begin{Bmatrix}\alpha\sigma\\ \nu_i\end{Bmatrix} A_\mu^{\nu_1,\cdots,\nu_{i-1},\alpha,\nu_{i+1},\cdots,\nu_n} B_\rho^\tau \\ &+ \sum_{i=1}^t \sum_\alpha \begin{Bmatrix}\alpha\sigma\\ \tau_i\end{Bmatrix} B_\rho^{\tau_1,\cdots,\tau_{i-1},\alpha,\tau_{i+1},\cdots,\tau_t} A_\mu^\nu \\ &= B_\rho^\tau\frac{\mathfrak{D}A_\mu^\nu}{\mathfrak{D}x_\sigma} + A_\mu^\nu\frac{\mathfrak{D}B_\rho^\tau}{\mathfrak{D}x_\sigma}.\end{aligned}$$

Consider now the covariant derivative of the general symmetric covariant tensor of valence two, $g_{(2)}$, and of its reciprocal $g^{(2)}$.

$$\frac{\mathfrak{D}g_{\mu\nu}}{\mathfrak{D}x_\sigma} = \frac{\partial g_{\mu\nu}}{\partial x_\sigma} - \sum_\alpha \begin{Bmatrix}\mu\sigma\\ \alpha\end{Bmatrix} g_{\alpha\nu} - \sum_\alpha \begin{Bmatrix}\nu\sigma\\ \alpha\end{Bmatrix} g_{\mu\alpha}.$$

By (20) this becomes

$$\frac{\mathfrak{D}g_{\mu\nu}}{\mathfrak{D}x_\sigma} = \frac{\partial g_{\mu\nu}}{\partial x_\sigma} - \begin{bmatrix}\mu\sigma\\ \nu\end{bmatrix} - \begin{bmatrix}\nu\sigma\\ \mu\end{bmatrix},$$

which by (19) is identically zero. We therefore have

$$\frac{\mathfrak{D}g_{\mu\nu}}{\mathfrak{D}x_\sigma} = 0.$$

Consider now

$$(38) \qquad \sum_\nu g_{\mu\nu} g^{\nu\lambda} = g_\mu^\lambda = I.$$

We have

$$(39) \qquad \frac{\mathfrak{D}\left(\sum_\nu g_{\mu\nu} g^{\nu\lambda}\right)}{\mathfrak{D}x_\sigma} = \sum_\nu g_{\mu\nu} \frac{\mathfrak{D}g^{\nu\lambda}}{\mathfrak{D}x_\sigma} + \sum_\nu g^{\nu\lambda} \frac{\mathfrak{D}g_{\mu\nu}}{\mathfrak{D}x_\sigma} = \sum_\nu g_{\mu\nu} \frac{\mathfrak{D}g^{\nu\lambda}}{\mathfrak{D}x_\sigma}.$$

On the other hand by (34)

$$\frac{\mathfrak{D}g_\mu^\lambda}{\mathfrak{D}x_\sigma} = \frac{\partial g_\mu^\lambda}{\partial x_\sigma} - \sum_\alpha \begin{Bmatrix} \mu\,\sigma \\ \alpha \end{Bmatrix} g_\alpha^\lambda + \sum_\alpha \begin{Bmatrix} \alpha\,\sigma \\ \lambda \end{Bmatrix} g_\mu^\alpha.$$

Now

$$\frac{\partial g_\mu^\lambda}{\partial x_\sigma} = \frac{\partial I}{\partial x_\sigma} = 0,$$

$$\sum_\alpha \begin{Bmatrix} \mu\,\sigma \\ \alpha \end{Bmatrix} g_\alpha^\lambda = \begin{Bmatrix} \mu\,\sigma \\ \lambda \end{Bmatrix}; \qquad \sum_\alpha \begin{Bmatrix} \alpha\sigma \\ \lambda \end{Bmatrix} g_\mu^\alpha = \begin{Bmatrix} \mu\,\sigma \\ \lambda \end{Bmatrix}.$$

Hence

$$\frac{\mathfrak{D}g_\mu^\lambda}{\mathfrak{D}x_\sigma} = 0.$$

Therefore, in view of (38) we have from (39) that

$$\sum_\nu g_{\mu\nu} \frac{\mathfrak{D}g^{\nu\lambda}}{\mathfrak{D}x_\sigma} = 0,$$

from which it follows from §10 that

$$\frac{\mathfrak{D}g^{\nu\lambda}}{\mathfrak{D}x_\sigma} = 0.$$

We now have the results

$$(40) \qquad \frac{\mathfrak{D}g_{\mu\nu}}{\mathfrak{D}x_\sigma} = \frac{\mathfrak{D}g^{\mu\nu}}{\mathfrak{D}x_\sigma} = \frac{\mathfrak{D}g_\mu^\nu}{\mathfrak{D}x_\sigma} = 0.$$

By (37)

$$\frac{\mathfrak{D}(\sum_{\nu_1} g_{\lambda\nu_1} A_{\mu_1, \cdots, \mu_m}^{\nu_1, \cdots, \nu_n})}{\mathfrak{D}x_\sigma} = \sum_{\nu_1} g_{\lambda\nu_1} \frac{\mathfrak{D}A_\mu^\nu}{\mathfrak{D}x_\sigma} + \sum_{\nu_1} A_\mu^\nu \frac{\mathfrak{D}g_{\lambda\nu_1}}{\mathfrak{D}x_\sigma}.$$

The second term on the right is zero by (40). We also have

$$\frac{\mathfrak{D}(\sum_{\mu_1} g^{\lambda\mu_1}A^{\nu_1,\cdots,\nu_n}_{\mu_1,\cdots,\mu_m})}{\mathfrak{D}x_\sigma} = \sum_{\mu_1} g^{\lambda\mu_1}\frac{\mathfrak{D}A^{\nu}_{\mu}}{\mathfrak{D}x_\sigma} + \sum_{\mu_1} A^{\nu}_{\mu}\frac{\mathfrak{D}g^{\lambda\mu_1}}{\mathfrak{D}x_\sigma},$$

in which similarly the second term on the right is zero.

Hence for every tensor the operations of association and covariant differentiation are commutative.

§18. *The quadratic differential form. Fundamental tensors.*

It follows from (2) that $dx(=dx_1, \cdots, dx_4)$ is a contravariant tensor of valence one. Consider now the quadratic differential form

$$\sum_{\mu,\nu} g_{\mu\nu}dx_\mu dx_\nu,$$

$g_{\mu\nu}$ being the symmetric covariant tensor of valence two which we have been studying. It follows from § 7 that this function is an invariant; this quantity we shall call

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu}dx_\mu dx_\nu. \tag{41}$$

It should be noted that the equation (22) of transformation of $g_{\mu\nu}$ is the same that we should have if we took (41) as the fundamental form of the tensor theory and applied (T) to it as we did to (1) in § 1.

In the Theory of Relativity the form (41) defines the geometry of the 4-dimensional space $x_1, \cdots, x_4$ in the sense of the theory of spaces as developed by Gauss and Riemann, in which the properties of a space are equivalent to the properties of a symmetric quadratic differential form. Thus ds^2 is the square of the differential element of length ds; and the length s of a path C from a point P_1 of the space to a point P_2 of the space is

$$\int_{P_1}^{P_2} ds.$$

The existence of this integral will not be discussed here.

A *fundamental tensor* is defined as a tensor which is a function of only the $g_{\mu\nu}$ and their derivatives. Examples of fundamental tensors are the following:

g_2	$=g_{\mu\nu}$	covariant of valence two
g^{-1}	$=g^{\mu\nu}$	contravariant of valence two
I	$=g^{\nu}_{\mu}$	invariant, and also mixed of type $\binom{1}{1}$
ds^2		invariant
$R^{\sigma}_{\mu\nu\lambda}$		of type $\binom{1}{3}$

Other tensors are

dx contravariant of valence one

h_μ covariant of valence one

We can introduce two theorems that enable us to form new tensors from given ones.

I. If $A(h_i)$ is an invariant then dA is an invariant.

For $A'=A$ implies that $dA'=dA$.

II. If $A^{(0)}_{(0)}(h_i)$ is an invariant then $\partial A/\partial x_\mu$ is a covariant of valence one. For

$$\frac{\partial A'}{\partial x'_\mu}=\frac{\partial A}{\partial x'_\mu},$$

since $A'=A$. Therefore

$$\frac{\partial A'}{\partial x'_\mu}=\frac{\partial A}{\partial x'_\mu}=\sum_i\frac{\partial x_i}{\partial x'_\mu}\,\frac{\partial A}{\partial x_i}.$$

But by § 2

$$\left(\frac{\partial A}{\partial x_\mu}\right)'=\frac{\partial A'}{\partial x'_\mu}.$$

Therefore

$$\left(\frac{\partial A}{\partial x_\mu}\right)'=\sum_i\frac{\partial x_i}{\partial x'_\mu}\,\frac{\partial A}{\partial x_i},$$

which shows that $\partial A/\partial x_\mu$ is a covariant of valence one.

§19. *Fundamental tensors.*

It was stated in § 6 that our object is to find covariant equations of (1), and that this is to be accomplished by finding examples of tensors. The problem can now be made more definite by limiting the tensors sought for to fundamental tensors.

It is to be noted that the results of § 15 enable us to form covariant partial differential equations by equating to zero the covariant derivative of a tensor; but we cannot obtain such equations by covariant differentiation of the fundamental tensor $g_{\mu\nu}$, since by (40) the covariant derivatives of $g_{\mu\nu}$ are identically zero.

§20. *Additional properties of $g_{\mu\nu}$.*

We shall consider in this section some additional differential properties of $g_{\mu\nu}$. Now $g^{\mu\nu}=1/g$ adjoint $g_{\mu\nu}$, and therefore adj $g_{\mu\nu}=gg^{\mu\nu}$. Let us expand g according to the elements of the μth row:

$$g=g_{\mu\nu}\text{ adj }g_{\mu\nu}+\text{terms free of }g_{\mu\nu}.$$

We note that adj $g_{\mu\nu}$ is free of $g_{\mu\nu}$. Hence

$$dg=\text{adj }g_{\mu\nu}\,dg_{\mu\nu}+\text{terms free of }dg_{\mu\nu}.$$

Since this is true for all values of μ and ν we have

$$dg = \sum_{\mu,\nu} \text{adj } g_{\mu\nu} dg_{\mu\nu}.$$

Also

$$\sum_{\mu,\nu} g^{\mu\nu} dg_{\mu\nu} = \sum_{\mu,\nu} \frac{\text{adj } g_{\mu\nu}}{g} dg_{\mu\nu} = \frac{dg}{g}.$$

Replacing g by g^{-1} in this we have

$$\sum_{\mu,\nu} g_{\mu\nu} dg^{\mu\nu} = \frac{dg^{-1}}{g^{-1}} = -\frac{dg}{g}.$$

Hence

$$\frac{dg}{g} = \sum_{\mu,\nu} g^{\mu\nu} dg_{\mu\nu} = -\sum_{\mu,\nu} g_{\mu\nu} dg^{\mu\nu}. \tag{42}$$

We also have

$$\sum_{\beta} g_{\alpha\beta} g^{\beta\gamma} = I(\alpha,\gamma).$$

Hence

$$\sum_{\beta} g_{\alpha\beta} dg^{\beta\gamma} = -\sum_{\beta} g^{\beta\gamma} dg_{\alpha\beta}.$$

Multiplying by $g_{\gamma\delta}$ and summing as to γ,

$$\sum_{\beta,\gamma} g_{\alpha\beta} g_{\gamma\delta} dg^{\beta\gamma} = -\sum_{\beta,\gamma} g^{\beta\gamma} g_{\gamma\delta} dg_{\alpha\beta} = -\sum_{\beta} I(\beta,\delta) dg_{\alpha\beta} = -dg_{\alpha\delta}.$$

Let $B^{(2)}$ be a symmetric tensor; then

$$-\sum_{\alpha,\delta} B^{\alpha\delta} dg_{\alpha\delta} = \sum_{\beta,\gamma} \left(\sum_{\alpha,\delta} g_{\gamma\delta} g_{\alpha\beta} B^{\alpha\delta} \right) dg^{\beta\gamma} = \sum_{\beta,\gamma} B_{\beta\gamma} dg^{\beta\gamma},$$

where $B_{(2)}$ is defined as the associated tensor of $B^{(2)}$. Hence

$$\sum_{\mu,\nu} B^{\mu\nu} dg_{\mu\nu} = -\sum_{\mu,\nu} B_{\mu\nu} dg^{\mu\nu}.$$

In developing the formulae obtained in this section partial differentiation with respect to x_λ may be substituted for the differential operation. We thus have

$$\frac{1}{g}\frac{\partial g}{\partial x_\lambda} = \sum_{\mu,\nu} g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x_\lambda} = -\sum_{\mu,\nu} g_{\mu\nu} \frac{\partial g^{\mu\nu}}{\partial x_\lambda};$$

$$\sum_{\mu,\nu} B^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x_\lambda} = -\sum_{\mu,\nu} B_{\mu\nu} \frac{\partial g^{\mu\nu}}{\partial x_\lambda}.$$

We obtain another relation as follows: by (18)

$$\sum_{\mu} \left\{ {\lambda\mu \atop \mu} \right\} = \frac{1}{2} \sum_{\mu,\nu} g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x_\lambda}.$$

Hence

$$(43) \qquad \sum_{\mu} \left\{ {\lambda\mu \atop \mu} \right\} = \frac{1}{2g} \frac{\partial g}{\partial x_\lambda} = \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial x_\lambda} = \frac{\partial}{\partial x_\lambda} \log \sqrt{-g},$$

where we have used $-g$ because, in fact, g is always negative in the physical applications.

§21. *Divergence of a tensor.*

By a *divergence* of a tensor $A^{\nu_1,\ldots,\nu_n}_{\mu_1,\ldots,\mu_m}(h_\mu)$ we mean a tensor of type $\binom{n-1}{m}$ derived from A by contracting its covariant derivative with respect to the index of differentiation:

$$\nabla_{\nu_i} A^{\nu_1,\ldots,\nu_n}_{\mu_1,\ldots,\mu_m} \equiv \sum_{\nu_i} \frac{\mathfrak{D} A^{\nu_1,\ldots,\nu_n}_{\mu_1,\ldots,\mu_m}}{\mathfrak{D} x_{\nu_i}}.$$

Clearly there are n such divergences of A. For example the divergence of the tensor A^ν is by (34) and (43)

$$\nabla_\nu A^\nu = \sum_\nu \left[\frac{\partial A^\nu}{\partial x_\nu} + \sum_\alpha \left\{ {\alpha\, \nu \atop \nu} \right\} A^\alpha \right]$$

$$= \sum_\nu \frac{\partial A^\nu}{\partial x_\nu} + \frac{1}{\sqrt{-g}} \sum_\alpha A^\alpha \frac{\partial}{\partial x_\alpha} \sqrt{-g} = \frac{1}{\sqrt{-g}} \sum_\nu \frac{\partial}{\partial x_\nu} (A^\nu \sqrt{-g}),$$

which is an invariant. Similarly

$$\nabla_\nu A^\nu_\mu = \sum_\nu \left[\frac{\partial A^\nu_\mu}{\partial x_\nu} - \sum_\alpha \left\{ {\mu\nu \atop \alpha} \right\} A^\nu_\alpha + \sum_\alpha \left\{ {\alpha\, \nu \atop \nu} \right\} A^\alpha_\mu \right]$$

$$= \frac{1}{\sqrt{-g}} \sum_\nu \frac{\partial}{\partial x_\nu} (A^\nu_\mu \sqrt{-g}) - \sum_{\nu,\alpha} A^\nu_\alpha \left\{ {\mu\nu \atop \alpha} \right\}.$$

By (17) the last term is

$$- \sum_{\nu,\alpha} A^\nu_\alpha \sum_\beta g^{\alpha\beta} \begin{bmatrix} \mu\nu \\ \beta \end{bmatrix} = -\frac{1}{2} \sum_{\nu,\beta} \left(\frac{\partial g_{\mu\beta}}{\partial x_\nu} + \frac{\partial g_{\nu\beta}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\beta} \right) \sum_\alpha g^{\alpha\beta} A^\nu_\alpha.$$

If $\sum_\alpha g^{\alpha\beta} A_\alpha^\nu$ is symmetric this term reduces to

$$-\frac{1}{2}\sum_{\nu,\alpha,\beta}\frac{\partial g_{\nu\beta}}{\partial x_\mu}\, g^{\alpha\beta} A_\alpha^\nu,$$

and we have

$$\nabla_\nu A_\mu^\nu = \frac{1}{\sqrt{-g}}\sum_\nu \frac{\partial}{\partial x_\nu}(A_\mu^\nu\sqrt{-g}) - \frac{1}{2}\sum_{\nu,\alpha,\beta}\frac{\partial g_{\nu\beta}}{\partial x_\mu} g^{\alpha\beta} A_\alpha^\nu.$$

By § 20 we may write this

$$\nabla_\nu A_\mu^\nu = \frac{1}{\sqrt{-g}}\sum_\nu \frac{\partial}{\partial x_\nu}(A_\mu^\nu\sqrt{-g}) + \frac{1}{2}\sum_{\nu,\alpha,\beta}\frac{\partial g^{\nu\beta}}{\partial x_\mu} g_{\alpha\beta} A_\nu^\alpha.$$

CHAPTER IV

THE TENSORS $R_{\rho\mu\nu\lambda}$ AND $G_{\mu\nu}$.

§22. *Flat space.*

A space S defined by the form

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu} dx_\mu dx_\nu, \tag{41}$$

where the $g_{\mu\nu}$ are given functions of the x_i, is called *flat* if (41) is equivalent to a form with constant coefficients.

A necessary and sufficient condition that S be flat is

$$R^{\rho}_{\mu\nu\lambda} = 0.$$

That the condition is necessary is evident at once. For by hypothesis there exists a (T) such that the $g'_{\mu\nu}$ are constants. Hence by (16) and (17) $\left\{ {\mu\nu \atop \sigma} \right\} = 0$. Hence by (28) $R'^{\rho}_{\mu\nu\lambda} = 0$. But since $R^{\rho}_{\mu\nu\lambda}$ is a tensor it follows that $R^{\rho}_{\mu\nu\lambda} = 0$.

That the condition is sufficient will be proved in § 24.

§23. *Pairs of quadratic differential forms.*

We shall state and prove two theorems. Consider two quadratic differential forms

$$\sum_{\mu,\nu} g_{\mu\nu} dx_\mu dx_\nu, \tag{41}$$

and

$$\sum_{\mu,\nu} \bar{g}_{\mu\nu} d\bar{x}_\mu d\bar{x}_\nu,$$

and a non-singular transformation

$$x_i = x_i(\bar{x}_1, \cdots, \bar{x}_4). \tag{U}$$

Let us assume that

$$\frac{\partial^2 x_e}{\partial \bar{x}_\mu \partial \bar{x}_\nu} = \sum_\rho \frac{\partial x_e}{\partial \bar{x}_\rho} \left\{ {\mu\nu \atop \rho} \right\}_{\bar{g}(2)} - \sum_{i,j} \frac{\partial x_i}{\partial \bar{x}_\mu} \frac{\partial x_j}{\partial \bar{x}_\nu} \left\{ {ij \atop e} \right\}. \tag{44}$$

Let (U) transform (41) into

$$\sum_{\mu,\nu} \bar{f}_{\mu\nu} d\bar{x}_\mu d\bar{x}_\nu.$$

Let $\bar{g}_{\mu\nu}-\bar{f}_{\mu\nu}=\bar{k}_{\mu\nu}$; then

$$\bar{g}_{\mu\nu} = \sum_{i,j} \frac{\partial x_i}{\partial \bar{x}_\mu} \frac{\partial x_j}{\partial \bar{x}_\nu} g_{ij} + \bar{k}_{\mu\nu}.$$

Now by the method of § 13 we find that

$$\frac{\partial^2 x_e}{\partial \bar{x}_\mu \partial \bar{x}_\nu} = \sum_\rho \frac{\partial x_e}{\partial \bar{x}_\rho} \begin{Bmatrix} \mu\nu \\ \rho \end{Bmatrix}_{\bar{g}_{(2)}} - \sum_{i,j} \frac{\partial x_i}{\partial \bar{x}_\mu} \frac{\partial x_j}{\partial \bar{x}_\nu} \begin{Bmatrix} ij \\ e \end{Bmatrix}$$

$$- \sum_{\rho,\sigma} \frac{\partial x_e}{\partial \bar{x}_\rho} \bar{g}^{\sigma\rho} \begin{bmatrix} \mu\nu \\ \sigma \end{bmatrix}_{\bar{k}_{(2)}}.$$

Hence by (44)

$$\sum_{\rho,\sigma} \frac{\partial x_e}{\partial \bar{x}_\rho} \bar{g}^{\sigma\rho} \begin{bmatrix} \mu\nu \\ \sigma \end{bmatrix}_{\bar{k}_{(2)}} = 0,$$

so that $\left[{\mu\nu \atop \sigma}\right]_{\bar{k}_{(2)}}=0$, and therefore, by § 12, $\bar{k}_{\mu\nu}=$ constant. Thus $\bar{g}_{\mu\nu}=\bar{f}_{\mu\nu}+$constant, which is the first theorem.

Consider now two forms

$$\sum_{\mu,\nu} \bar{g}_{\mu\nu} d\bar{x}_\mu d\bar{x}_\nu,$$

and

(45) $$\sum_{\mu,\nu} \bar{f}_{\mu\nu} d\bar{x}_\mu d\bar{x}_\nu,$$

and suppose that $\left\{{\mu\nu \atop \rho}\right\}_{\bar{g}_{(2)}}=\left\{{\mu\nu \atop \rho}\right\}_{\bar{f}_{(2)}}$. Let (U^{-1}) transform (45) into

(41) $$\sum_{\mu,\nu} g_{\mu\nu} dx_\mu dx_\nu.$$

Then (U) transforms (41) into (45). By (25) we have

$$\frac{\partial^2 x_e}{\partial \bar{x}_\mu \partial \bar{x}_\nu} = \sum_\rho \frac{\partial x_e}{\partial \bar{x}_\rho} \begin{Bmatrix} \mu\nu \\ \rho \end{Bmatrix}_{\bar{f}_{(2)}} - \sum_{i,j} \frac{\partial x_i}{\partial \bar{x}_\mu} \frac{\partial x_j}{\partial \bar{x}_\nu} \begin{Bmatrix} ij \\ e \end{Bmatrix}.$$

But since $\left\{{\mu\nu \atop \rho}\right\}_{\bar{f}_{(2)}}=\left\{{\mu\nu \atop \rho}\right\}_{\bar{g}_{(2)}}$ this equation becomes exactly (44). Hence by the preceding theorem

$$\bar{g}_{\mu\nu}=\bar{f}_{\mu\nu}+\text{constant}.$$

Thus $\left\{{\mu\nu \atop \rho}\right\}_{f_{(2)}} = \left\{{\mu\nu \atop \rho}\right\}_{g_{(2)}}$ implies $g_{\mu\nu} = f_{\mu\nu}+$constant; the converse is clearly not true.

§24. *Equivalence of quadratic differential forms.*

Let us determine the condition that two symmetric quadratic differential forms

(46)
$$\sum_{\mu,\nu} g_{\mu\nu} dx_\mu dx_\nu, \quad \det g_{\mu\nu} \neq 0\,;$$
$$\sum \bar{a}_{\mu\nu} d\bar{x}_\mu d\bar{x}_\nu, \quad \bar{a}_{\mu\nu} = \text{const.}, \ \det \bar{a}_{\mu\nu} \neq 0,$$

be *equivalent*, that is, that there exist a non-singular transformation connecting the two forms.†

If the forms are equivalent there exists a non-singular transformation (U) which transforms the first into the second. Then by (25)

$$\frac{\partial^2 x_e}{\partial \bar{x}_\mu \partial \bar{x}_\nu} = \sum_\rho \frac{\partial x_e}{\partial \bar{x}_\rho}\left\{ {\mu\nu \atop \rho} \right\}_{\bar{a}_{(2)}} - \sum_{i,j} \frac{\partial x_i}{\partial \bar{x}_\mu}\,\frac{\partial x_j}{\partial \bar{x}_\nu}\left\{ {ij \atop e} \right\} = - \sum_{i,j} \frac{\partial x_i}{\partial \bar{x}_\mu}\,\frac{\partial x_j}{\partial \bar{x}_\nu}\left\{ {ij \atop e} \right\} \tag{47}$$

since $\left\{ {\mu\nu \atop \rho} \right\}_{\bar{a}_{(2)}} = 0$. Consider (47) as first order partial differential equations in the unknowns $\partial x_i/\partial \bar{x}_j$ and independent variables x_i. By a well-known theorem‡ we know that if (47) is a *complete system* there exist 16 quantities $\partial x_i/\partial \bar{x}_j$ of $\det \neq 0$ satisfying (47); these 16 quantities furnish by direct quadrature a non-singular transformation (W). A sufficient condition that (47) be a complete system is

$$\frac{\partial^3 x_e}{\partial \bar{x}_\lambda \partial \bar{x}_\nu \partial \bar{x}_\mu} - \frac{\partial^3 x_e}{\partial \bar{x}_\nu \partial \bar{x}_\lambda \partial \bar{x}_\mu} = 0.$$

By (29) this becomes.

$$\sum_{i,j,k} \frac{\partial x_i}{\partial \bar{x}_\mu}\,\frac{\partial x_j}{\partial \bar{x}_\nu}\,\frac{\partial x_k}{\partial \bar{x}_\lambda} R^e_{ijk} = 0.$$

Thus $R^\rho_{\mu\nu\lambda} = 0$ is a sufficient condition that (47) be a complete system and hence that there exist a non-singular (W) satisfying (47). Let this (W) transform the first form of (46) into

$$\sum_{\mu,\nu} \bar{f}_{\mu\nu} d\bar{x}_\mu d\bar{x}_\nu .$$

Now we may apply the first theorem of §23 with $\bar{g}_{\mu\nu} = \bar{a}_{\mu\nu}$, so that $\bar{a}_{\mu\nu} - \bar{f}_{\mu\nu} = \text{const.}$ Hence $\bar{f}_{\mu\nu} = \text{constant}$, and the existing non-singular transformation (W) transforms the first of (46) into a form with constant coefficients.

Hence $R^\rho_{\mu\nu\lambda} = 0$ is a sufficient condition that (41) be equivalent to a quadratic differential form with constant coefficients.

§25. *The quadratic differential form of flat space.*

In the case that $R^\rho_{\mu\nu\lambda} = 0$ we may particularize still further the form to which the coefficients $g_{\mu\nu}$ may be reduced. For by a well-known theorem every quadratic differential form with constant coefficients is equivalent under a real non-singular transformation to one in which only the square

† The general situation in which the $\bar{a}_{\mu\nu}$ are not constants is of no interest to us here. For this case see Veblen: Invariants of Quadratic Differential Forms. Cambridge Tracts; 1927, Chap. V, §§6–8.

‡ Goursat: *Les Equations aux Dérivées Partiëlles du Premier Ordre*, 1922, p. 116.

terms are present, the coefficients being numerically equal to one. If we start with real $g_{\mu\nu}$ there are five forms to which this reduction may lead:

$$(a)\qquad ds^2 = +\,dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

$$(b)\qquad ds^2 = -\,dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

$$(c)\qquad ds^2 = -\,dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2$$

$$(d)\qquad ds^2 = -\,dx_1^2 - dx_2^2 - dx_3^2 + dx_4^2$$

$$(e)\qquad ds^2 = -\,dx_1^2 - dx_2^2 - dx_3^2 - dx_4^2.$$

It is clear that these forms are all equivalent under purely imaginary transformations, and hence that no two of them are equivalent under a real transformation. For suppose that a transformation (T) transforms, for example, (*b*) into (*a*); then since the imaginary transformation (Q) $\left\{\begin{smallmatrix} x_1 = ix_1' \\ x_\nu = x_\nu'(\nu\neq 1)\end{smallmatrix}\right.$ transforms (*a*) into (*b*) we see that the transformation (TQ) transforms (*a*) into itself, and hence (TQ)$=I$. Hence T$=$Q^{-1}, and T is imaginary. Which one of these forms a given ds^2 is equivalent to under real transformation depends entirely on the $g_{\mu\nu}$.

We may mention in passing that the fourth of the forms is the one assumed to apply to real space-time systems in the Special Theory of Relativity.

§26. *The covariant Riemann-Christoffel tensor.*

Let us consider the associated tensor

$$(48)\qquad R_{\rho\mu\nu\lambda} = \sum_\beta g_{\rho\beta} R^{\beta}_{\mu\nu\lambda},$$

which we shall call the *covariant Riemann-Christoffel Tensor.* By (28) and (20) we find that

$$R_{\rho\mu\nu\lambda} = \sum_\alpha \begin{Bmatrix}\mu\lambda\\ \alpha\end{Bmatrix}\begin{bmatrix}\alpha\nu\\ \rho\end{bmatrix} - \sum_\alpha \begin{Bmatrix}\mu\nu\\ \alpha\end{Bmatrix}\begin{bmatrix}\alpha\lambda\\ \rho\end{bmatrix} + \frac{\partial}{\partial x_\nu}\begin{bmatrix}\mu\lambda\\ \rho\end{bmatrix} - \frac{\partial}{\partial x_\lambda}\begin{bmatrix}\mu\nu\\ \rho\end{bmatrix}$$
$$- \sum_\alpha \begin{Bmatrix}\mu\lambda\\ \alpha\end{Bmatrix}\frac{\partial g_{\rho\alpha}}{\partial x_\nu} + \sum_\alpha \begin{Bmatrix}\mu\nu\\ \alpha\end{Bmatrix}\frac{\partial g_{\rho\alpha}}{\partial x_\lambda}.$$

By (19) and (16)

$$(49)\qquad R_{\rho\mu\nu\lambda} = -\sum_\alpha \begin{Bmatrix}\mu\lambda\\ \alpha\end{Bmatrix}\begin{bmatrix}\rho\nu\\ \alpha\end{bmatrix} + \sum_\alpha \begin{Bmatrix}\mu\nu\\ \alpha\end{Bmatrix}\begin{bmatrix}\rho\lambda\\ \alpha\end{bmatrix}$$
$$+\frac{1}{2}\left(\frac{\partial^2 g_{\rho\lambda}}{\partial x_\mu \partial x_\nu} + \frac{\partial^2 g_{\mu\nu}}{\partial x_\rho \partial x_\lambda} - \frac{\partial^2 g_{\mu\lambda}}{\partial x_\nu \partial x_\rho} - \frac{\partial^2 g_{\nu\rho}}{\partial x_\mu \partial x_\lambda}\right).$$

We see immediately that

$$(50) \qquad \begin{cases} R_{\rho\mu\nu\lambda} = -R_{\rho\mu\lambda\nu}, \\ R_{\rho\mu\nu\lambda} = R_{\nu\lambda\rho\mu}\,; \text{ hence } R_{\rho\mu\nu\lambda} = -R_{\mu\rho\nu\lambda}, \\ R_{\rho\mu\nu\lambda} + R_{\rho\nu\lambda\mu} + R_{\rho\lambda\mu\nu} = 0. \end{cases}$$

The relation, $R_{\rho\mu\nu\lambda} = -R_{\mu\rho\nu\lambda}$, follows from the first two. The first two of these reduce the number of linearly independent components of $R_{\rho\mu\nu\lambda}$ from 256 to 21; the third relation still further reduces the number to 20. Hence this tensor has not more than 20 linearly independent components.

A set of 20 linearly independent components of this tensor is indicated by the following set of indices:

1212	1234	1334	2324
1213	1313	1414	2334
1214	1314	1424	2424
1223	1323	1434	2434
1224	1324	2323	3434.

These 20 components are clearly linearly independent, for by (49) each one contains a second derivative which none of the others contain; e.g. the only one containing $\partial^2 g_{13}/\partial x_2 \partial x_4$ is R_{1234}, the only one containing $\partial^2 g_{12}/\partial x_3 \partial x_4$ is R_{1324}, the only one containing $\partial^2 g_{12}/\partial x_1 \partial x_2$ is R_{1212}, etc.

From § 11 it follows that $R^{\rho}_{\mu\nu\lambda} = 0$ and $R_{\rho\mu\nu\lambda} = 0$ are equivalent equations. Hence by § 24: *A necessary and sufficient condition that S be flat is* $R_{\rho\mu\nu\lambda} = 0$.

§27. *The Ricci-Einstein tensor.*

Let us consider the following tensor

$$(51) \qquad G_{\mu\nu} = \sum_{\rho,\lambda} g^{\rho\lambda} R_{\rho\mu\nu\lambda} = \sum_{\rho,\lambda} g^{\rho\lambda} \sum_{\beta} g_{\rho\beta} R^{\beta}{}_{\mu\nu\lambda} = \sum_{\beta,\lambda} R^{\beta}{}_{\mu\nu\lambda} \sum_{\rho} g^{\rho\lambda} g_{\rho\beta} = \sum_{\lambda} R^{\lambda}{}_{\mu\nu\lambda}.$$

By (28) this is,

$$G_{\mu\nu} = \sum_{\alpha,\lambda} \begin{Bmatrix} \mu\lambda \\ \alpha \end{Bmatrix} \begin{Bmatrix} \alpha\nu \\ \lambda \end{Bmatrix} - \sum_{\alpha,\lambda} \begin{Bmatrix} \mu\nu \\ \alpha \end{Bmatrix} \begin{Bmatrix} \alpha\lambda \\ \lambda \end{Bmatrix} + \sum_{\lambda} \frac{\partial}{\partial x_\nu} \begin{Bmatrix} \mu\lambda \\ \lambda \end{Bmatrix} - \sum_{\lambda} \frac{\partial}{\partial x_\lambda} \begin{Bmatrix} \mu\nu \\ \lambda \end{Bmatrix}.$$

By (43) we have

$$\sum_{\lambda} \begin{Bmatrix} \mu\lambda \\ \lambda \end{Bmatrix} = \frac{\partial}{\partial x_\mu} \log \sqrt{-g}\,;$$

hence

$$(52) \quad G_{\mu\nu} = -\sum_{\alpha} \frac{\partial}{\partial x_\alpha} \begin{Bmatrix} \mu\nu \\ \alpha \end{Bmatrix} + \sum_{\alpha,\beta} \begin{Bmatrix} \mu\alpha \\ \beta \end{Bmatrix} \begin{Bmatrix} \nu\beta \\ \alpha \end{Bmatrix} + \frac{\partial^2}{\partial x_\mu \partial x_\nu} \log \sqrt{-g} - \sum_{\alpha} \begin{Bmatrix} \mu\nu \\ \alpha \end{Bmatrix} \frac{\partial}{\partial x_\alpha} \log \sqrt{-g}.$$

Other tensors derived similarly from $R_{\rho\mu\nu\lambda}$ are

$$\sum_{\rho,\mu} g^{\rho\mu}R_{\rho\mu\nu\lambda}, \quad \sum_{\rho,\nu} g^{\rho\nu}R_{\rho\mu\nu\lambda}, \quad \sum_{\mu,\nu} g^{\mu\nu}R_{\rho\mu\nu\lambda}, \quad \sum_{\mu,\lambda} g^{\mu\lambda}R_{\rho\mu\nu\lambda}, \quad \sum_{\nu,\lambda} g^{\nu\lambda}R_{\rho\mu\nu\lambda}.$$

The first of these, $\sum_{\rho\mu} g^{\rho\mu}R_{\rho\mu\nu\lambda}$, is zero by the relation $R_{\rho\mu\nu\lambda} = -R_{\mu\rho\nu\lambda}$ of (50);

$$\sum_{\rho,\nu} g^{\rho\nu}R_{\rho\mu\nu\lambda} = -\sum_{\rho,\nu} g^{\rho\nu}R_{\rho\mu\lambda\nu} = -G_{\mu\lambda};$$

$$\sum_{\mu,\nu} g^{\mu\nu}R_{\rho\mu\nu\lambda} = -\sum_{\mu,\nu} g^{\mu\nu}R_{\mu\rho\nu\lambda} = \sum_{\mu,\nu} g^{\mu\nu}R_{\mu\rho\lambda\nu} = G_{\rho\lambda};$$

$$\sum_{\mu,\lambda} g^{\mu\lambda}R_{\rho\mu\nu\lambda} = -\sum_{\mu,\lambda} g^{\mu\lambda}R_{\mu\rho\nu\lambda} = -G_{\rho\nu};$$

$$\sum_{\nu,\lambda} g^{\nu\lambda}R_{\rho\mu\nu\lambda} = 0, \text{ since } R_{\rho\mu\nu\lambda} = -R_{\rho\mu\lambda\nu}.$$

Hence in this way the tensor $R_{\rho\mu\nu\lambda}$ yields only one new tensor, the *Ricci-Einstein tensor* $G_{\mu\nu}$, which is obviously symmetric by (52).

If we form the inner product (§ 7) of $g^{\mu\nu}$ and $G_{\mu\nu}$ we obtain an invariant

$$G = \sum_{\mu,\nu} g^{\mu\nu}G_{\mu\nu} = \sum_{\mu,\nu,\lambda,\rho} g^{\mu\nu}g^{\rho\lambda}R_{\rho\mu\nu\lambda}. \tag{53}$$

The equations $G_{\mu\nu}=0$ are Einstein's *field equations of gravitation* in free space; the quantity G is the Gauss-Riemann *invariant curvature* of space.

§28. *Canonical coördinates.*

Given

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu}dx_\mu dx_\nu,$$

we wish to show that there exists a transformation (T)† such that all 40 derivatives $\partial g'_{\mu\nu}/\partial x'_\sigma$ vanish at a previously chosen non-singular point P_0.‡

First, by a simple translation, let us move the origin to P_0. Now make the transformation

$$x_i = x'_i - \frac{1}{2}\sum_{\alpha,\beta}\left\{ {\alpha\beta \atop i} \right\}_0 x'_\alpha x'_\beta, \tag{T'}$$

where $\left\{ {\alpha\beta \atop i} \right\}_0$ is the value of $\left\{ {\alpha\beta \atop i} \right\}$ at the origin. Then

$$\begin{cases} \dfrac{\partial x_e}{\partial x'_\mu} = I(e,\mu) - \displaystyle\sum_\alpha \left\{ {\alpha\mu \atop e} \right\}_0 x'_\alpha, \\ \dfrac{\partial^2 x_e}{\partial x'_\mu \partial x'_\nu} = -\left\{ {\nu\mu \atop e} \right\}_0 = -\left\{ {\mu\nu \atop e} \right\}_0. \end{cases}$$

† Cf. Eddington: The Mathematical Theory of Relativity, ch. 3.
‡ A point at which $g=0$ we shall call a singular point.

Hence

$$\left(\det \frac{\partial x_i}{\partial x_j'}\right)_{x'=0} = 1.$$

Thus (T′) is non-singular, and has a unique inverse about the origin; hence x_i all zero implies x_i' all zero. Then by (25)

$$\sum_\rho \frac{\partial x_e}{\partial x_\rho'}\left\{\begin{matrix}\mu\nu\\ \rho\end{matrix}\right\}' - \sum_{i,j} \frac{\partial x_i}{\partial x_\mu'}\frac{\partial x_j}{\partial x_\nu'}\left\{\begin{matrix}ij\\ e\end{matrix}\right\} = -\left\{\begin{matrix}\mu\nu\\ e\end{matrix}\right\}_0;$$

at the origin this equation becomes

$$\sum_\rho I(e,\rho)\left\{\begin{matrix}\mu\nu\\ \rho\end{matrix}\right\}_0' - \sum_{i,j} I(i,\mu)I(j,\nu)\left\{\begin{matrix}ij\\ e\end{matrix}\right\}_0 = -\left\{\begin{matrix}\mu\nu\\ e\end{matrix}\right\}_0.$$

Hence

$$\left\{\begin{matrix}\mu\nu\\ e\end{matrix}\right\}_0' - \left\{\begin{matrix}\mu\nu\\ e\end{matrix}\right\}_0 = -\left\{\begin{matrix}\mu\nu\\ e\end{matrix}\right\}_0,$$

and thus

$$\left\{\begin{matrix}\mu\nu\\ e\end{matrix}\right\}_0' = 0.$$

Hence, by (20) and (19).

$$\left(\frac{\partial g'_{\mu\nu}}{\partial x_\sigma'}\right)_0 = 0,$$

as was to be shown.

If now we make the transformation

$$\text{(T}''\text{)} \qquad x_i' = x_i'' + \frac{1}{6}\sum_{\alpha,\beta,\gamma} a_i(\alpha,\beta,\gamma)x_\alpha''x_\beta''x_\gamma'',$$

where the $a_i(\alpha, \beta, \gamma)$ are constants completely symmetric in α, β, γ, then

$$(54) \qquad \begin{cases} \dfrac{\partial x_e'}{\partial x_\mu''} = I(e,\mu) + \dfrac{1}{2}\displaystyle\sum_{\alpha,\beta} a_e(\mu,\alpha,\beta)x_\alpha''x_\beta'', \\ \dfrac{\partial^2 x_e'}{\partial x_\mu''\partial x_\nu''} = \displaystyle\sum_\alpha a_e(\mu,\nu,\alpha)x_\alpha'', \\ \dfrac{\partial^3 x_e'}{\partial x_\mu''\partial x_\nu''\partial x_\sigma''} = a_e(\mu,\nu,\sigma). \end{cases}$$

Hence

$$\left(\det\frac{\partial x_i'}{\partial x_j''}\right)_{x''=0} = 1.$$

Thus (T″) is non-singular, and has a unique inverse about the origin; hence x_i' all zero implies x_i'' all zero. By (25)

$$(25)' \qquad \frac{\partial^2 x_e'}{\partial x_\mu''\partial x_\nu''} = \sum_\rho \frac{\partial x_e'}{\partial x_\rho''}\left\{\begin{matrix}\mu\nu\\ \rho\end{matrix}\right\}'' - \sum_{i,j}\frac{\partial x_i'}{\partial x_\mu''}\frac{\partial x_j'}{\partial x_\nu''}\left\{\begin{matrix}ij\\ e\end{matrix}\right\}'.$$

Hence

$$(55) \qquad \frac{\partial^3 x_e'}{\partial x_\mu''\partial x_\nu''\partial x_\sigma''} = \sum_\rho \frac{\partial^2 x_e'}{\partial x_\rho''\partial x_\sigma''}\left\{\begin{matrix}\mu\nu\\ \rho\end{matrix}\right\}'' + \sum_\rho \frac{\partial x_e'}{\partial x_\rho''}\frac{\partial}{\partial x_\sigma''}\left\{\begin{matrix}\mu\nu\\ \rho\end{matrix}\right\}''$$
$$- \sum_{i,j,k}\frac{\partial x_i'}{\partial x_\mu''}\frac{\partial x_j'}{\partial x_\nu''}\frac{\partial x_k'}{\partial x_\sigma''}\frac{\partial}{\partial x_k'}\left\{\begin{matrix}ij\\ e\end{matrix}\right\}' - \sum_{i,j}\frac{\partial x_i'}{\partial x_\mu''}\frac{\partial^2 x_j'}{\partial x_\nu''\partial x_\sigma''}\left\{\begin{matrix}ij\\ e\end{matrix}\right\}'$$
$$- \sum_{i,j}\frac{\partial^2 x_i'}{\partial x_\mu''\partial x_\sigma''}\frac{\partial x_j'}{\partial x_\nu''}\left\{\begin{matrix}ij\\ e\end{matrix}\right\}'.$$

By (54)

$$\left(\frac{\partial x_e'}{\partial x_\mu''}\right)_0 = I(e,\mu),$$

$$\left(\frac{\partial^2 x_e'}{\partial x_\mu''\partial x_\nu''}\right)_0 = 0,$$

and we have above that $\left\{\begin{smallmatrix}ij\\ e\end{smallmatrix}\right\}_0' = 0$. Hence (25)′ shows that

$$\sum_\rho I(e,\rho)\left\{\begin{matrix}\mu\nu\\ \rho\end{matrix}\right\}_0'' = \left\{\begin{matrix}\mu\nu\\ e\end{matrix}\right\}_0'' = 0.$$

Hence

$$\left(\frac{\partial g_{\mu\nu}''}{\partial x_\sigma''}\right)_0 = 0.$$

By (55) and (54)

$$a_e(\mu,\nu,\sigma) = \sum_\rho I(e,\rho)\left(\frac{\partial}{\partial x_\sigma''}\left\{\begin{matrix}\mu\nu\\ \rho\end{matrix}\right\}''\right)_0$$
$$- \sum_{i,j,k} I(i,\mu)I(j,\nu)I(k,\sigma)\left(\frac{\partial}{\partial x_k'}\left\{\begin{matrix}ij\\ e\end{matrix}\right\}'\right)_0.$$

Hence

$$(56)\quad \begin{cases} a_e(\mu,\nu,\sigma) = \left(\dfrac{\partial}{\partial x_\sigma''}\begin{Bmatrix}\mu\nu\\ e\end{Bmatrix}''\right)_0 - \left(\dfrac{\partial}{\partial x_\sigma}\begin{Bmatrix}\mu\nu\\ e\end{Bmatrix}'\right)_0, \\ a_e(\mu,\nu,\sigma) = \left(\dfrac{\partial}{\partial x_\nu''}\begin{Bmatrix}\mu\sigma\\ e\end{Bmatrix}''\right)_0 - \left(\dfrac{\partial}{\partial x_\nu'}\begin{Bmatrix}\mu\sigma\\ e\end{Bmatrix}'\right)_0, \\ a_e(\mu,\nu,\sigma) = \left(\dfrac{\partial}{\partial x_\mu''}\begin{Bmatrix}\nu\sigma\\ e\end{Bmatrix}''\right)_0 - \left(\dfrac{\partial}{\partial x_\mu'}\begin{Bmatrix}\nu\sigma\\ e\end{Bmatrix}'\right)_0, \end{cases}$$

where in the last two we have made cyclic changes in μ, ν, σ. Hence

$$\left(\frac{\partial}{\partial x_\sigma''}\begin{Bmatrix}\mu\nu\\ e\end{Bmatrix}''\right)_0 + \left(\frac{\partial}{\partial x_\nu''}\begin{Bmatrix}\mu\sigma\\ e\end{Bmatrix}''\right)_0 + \left(\frac{\partial}{\partial x_\mu''}\begin{Bmatrix}\nu\sigma\\ e\end{Bmatrix}''\right)_0$$
$$= \left(\frac{\partial}{\partial x_\sigma'}\begin{Bmatrix}\mu\nu\\ e\end{Bmatrix}'\right)_0 + \left(\frac{\partial}{\partial x_\nu'}\begin{Bmatrix}\mu\sigma\\ e\end{Bmatrix}'\right)_0 + \left(\frac{\partial}{\partial x_\mu'}\begin{Bmatrix}\nu\sigma\\ e\end{Bmatrix}'\right)_0 + 3a_e(\mu,\nu,\sigma).$$

Choose

$$a_e(\mu,\nu,\sigma,) = -\frac{1}{3}\left[\left(\frac{\partial}{\partial x_\sigma'}\begin{Bmatrix}\mu\nu\\ e\end{Bmatrix}'\right)_0 + \left(\frac{\partial}{\partial x_\nu'}\begin{Bmatrix}\mu\sigma\\ e\end{Bmatrix}'\right)_0 + \left(\frac{\partial}{\partial x_\mu'}\begin{Bmatrix}\nu\sigma\\ e\end{Bmatrix}'\right)_0\right];$$

we note that this choice is legitimate for it satisfies the requirements placed on the $a_e(\mu, \nu, \sigma)$. Then

$$\left(\frac{\partial}{\partial x_\sigma''}\begin{Bmatrix}\mu\nu\\ e\end{Bmatrix}''\right)_0 + \left(\frac{\partial}{\partial x_\nu''}\begin{Bmatrix}\mu\sigma\\ e\end{Bmatrix}''\right)_0 + \left(\frac{\partial}{\partial x_\mu''}\begin{Bmatrix}\nu\sigma\\ e\end{Bmatrix}''\right)_0 = 0,$$

and by § (26)

$$(57)\qquad \left(\frac{\partial}{\partial x_\nu''}\begin{bmatrix}\mu\sigma\\ e\end{bmatrix}''\right)_0 - \left(\frac{\partial}{\partial x_\sigma''}\begin{bmatrix}\mu\nu\\ e\end{bmatrix}''\right)_0 = (R''_{e\mu\nu\sigma})_0\,.$$

Hence, by the compound transformation (T″T′) we obtain a coördinate system in which

$$(58)\quad \begin{cases} \left(\dfrac{\partial g_{\mu\nu}}{\partial x_\sigma}\right)_0 = 0, \\ \left(\dfrac{\partial}{\partial x_\sigma}\begin{Bmatrix}\mu\nu\\ e\end{Bmatrix}\right)_0 + \left(\dfrac{\partial}{\partial x_\nu}\begin{Bmatrix}\mu\sigma\\ e\end{Bmatrix}\right)_0 + \left(\dfrac{\partial}{\partial x_\mu}\begin{Bmatrix}\nu\sigma\\ e\end{Bmatrix}\right)_0 = 0. \end{cases}$$

Coördinates for which (58) is true are called *canonical coördinates.*

Let us see what becomes of tensors when we transform to canonical coördinates. Choose an arbitrary non-singular point $P_0 = (x_i)_0$ in the x-system. Make the three non-singular transformations:

$$\mathrm{T}^0:\quad x_i = (x_i)_0 + x_i'\,,$$

$$(59)\qquad \mathrm{T}':\quad x_i' = x_i'' - \frac{1}{2}\sum_{\alpha,\beta}\begin{Bmatrix}\alpha\beta\\ i\end{Bmatrix}_0' x_\alpha'' x_\beta'',$$

$$T'': \quad x_i'' = x_i''' + \frac{1}{6}\sum_{\alpha,\beta,\gamma} a_i(\alpha,\beta,\gamma)x_\alpha''' x_\beta''' x_\gamma''',$$

where

$$a_e(\mu,\nu,\sigma) = -\frac{1}{3}\left[\left(\frac{\partial}{\partial x_\sigma''}\left\{ {\mu\nu \atop e} \right\}''\right)_0 + \left(\frac{\partial}{\partial x_\nu''}\left\{ {\mu\sigma \atop e} \right\}''\right)_0 + \left(\frac{\partial}{\partial x_\mu''}\left\{ {\nu\sigma \atop e} \right\}''\right)_0\right].$$

Since

$$\left(\frac{\partial x_i}{\partial x_j'}\right)_{P_0} = \left(\frac{\partial x_i'}{\partial x_j''}\right)_0 = \left(\frac{\partial x_i''}{\partial x_j'''}\right)_0 = I(i,j),$$

we have, for every tensor A,

$$(60) \qquad (A)_{P_0} = (A')_0 = (A'')_0 = (A''')_0.$$

§29. *Theorems on fundamental tensors.*

Consider a tensor

$$A\left(g_{\mu\nu}, \frac{\partial g_{\mu\nu}}{\partial x_\sigma}, \frac{\partial^2 g_{\mu\nu}}{\partial x_\rho \partial x_\sigma}\right).$$

Let us transform to canonical coördinates x_i' such that the point P_0 in the x_i transforms to the origin in the x_i'. Then, as we have just seen,

$$(A)_{P_0} = (A')_0 = A\left((g'_{\mu\nu})_0, \left(\frac{\partial g'_{\mu\nu}}{\partial x_\sigma'}\right)_0, \left(\frac{\partial^2 g'_{\mu\nu}}{\partial x_\rho' \partial x_\sigma'}\right)_0\right)$$

$$= A\left((g'_{\mu\nu})_0, 0, \left(\frac{\partial^2 g'_{\mu\nu}}{\partial x_\rho' \partial x_\sigma'}\right)_0\right),$$

since

$$\left(\frac{\partial g'_{\mu\nu}}{\partial x_\sigma'}\right)_0 = 0.$$

Now we have, by (57) and (58),

$$(61) \qquad \begin{aligned} &\left(\frac{\partial}{\partial x_\sigma'}\left\{ {\mu\nu \atop e} \right\}'\right)_0 + \left(\frac{\partial}{\partial x_\nu'}\left\{ {\mu\sigma \atop e} \right\}'\right)_0 + \left(\frac{\partial}{\partial x_\mu'}\left\{ {\nu\sigma \atop e} \right\}'\right)_0 = 0 \\ &\left(\frac{\partial}{\partial x_\nu'}\left[{\mu\sigma \atop e} \right]'\right)_0 - \left(\frac{\partial}{\partial x_\sigma'}\left[{\mu\nu \atop e} \right]'\right)_0 = (R'_{e\mu\nu\sigma})_0. \end{aligned}$$

We may consider (61) as linear algebraic equations in the 100 quantities $[\partial^2 g'_{\mu\nu}/\partial x_\rho' \partial x_\sigma']_0$; the coefficients are functions of $[(\partial g'_{\mu\nu})/(\partial x_\sigma')]_0 = 0$, $(g'^{\mu\nu})_0$, $[(\partial g'^{\mu\nu})/(\partial x_\sigma')]_0 = 0$, and hence of $(g'_{\mu\nu})_0$; the right members are linear functions of $(R'_{e\mu\nu\sigma})_0$. We know that these equations are consistent, for

they are satisfied by all $g_{\mu\nu}$. The first set of (61) is symmetric in μ, ν, σ and hence contains at most 80 linearly independent equations; the second set contains 20 linearly independent equations (§26); hence there are at most 100 linearly independent equations in (61).

If there were not 100 linearly independent equations (61), then, since the equations are consistent, we could choose at least one of the $[(\partial^2 g'_{\mu\nu})/(\partial x'_\rho \partial x'_\sigma)]_0$, for example $[(\partial^2 g'_{11})/(\partial x'^2_1)]_0$ in any way we wish, say equal to k. But since $(g_{11})_{P_0}$ is arbitrary in the x_i we cannot say that $[(\partial^2 g'_{11})/(\partial x'^2_1)]_0$ is equal to k.

Now we can solve the 100 linearly independent equations (61) for $[(\partial^2 g'_{\mu\nu})/(\partial x'_\rho \partial x'_\sigma)]_0$ as rational functions of $(g'_{\mu\nu})_0$ and $(R'_{e\mu\nu\sigma})_0$, linear in $(R'_{e\mu\nu\sigma})_0$.

The equations (61) can in fact be solved very easily. By (17) we may write the first of (61) as

$$\left(\frac{\partial}{\partial x'_\sigma} \sum_\alpha g'^{\alpha e}\begin{bmatrix}\mu\nu\\ \alpha\end{bmatrix}'\right)_0 + \left(\frac{\partial}{\partial x'_\nu} \sum_\alpha g'^{\alpha e}\begin{bmatrix}\mu\sigma\\ \alpha\end{bmatrix}'\right)_0 + \left(\frac{\partial}{\partial x'_\mu} \sum_\alpha g'^{\alpha e}\begin{bmatrix}\nu\sigma\\ \alpha\end{bmatrix}'\right)_0 = 0 .$$

If we multiply by $(g'_{\beta e})_0$ and sum as to e, noting that $(\partial g'^{\alpha e}/\partial x'_\rho)_0 = 0$, we have

$$\left(\frac{\partial}{\partial x'_\sigma}\begin{bmatrix}\mu\nu\\ \alpha\end{bmatrix}'\right)_0 + \left(\frac{\partial}{\partial x'_\nu}\begin{bmatrix}\mu\sigma\\ \alpha\end{bmatrix}'\right)_0 + \left(\frac{\partial}{\partial x'_\mu}\begin{bmatrix}\nu\sigma\\ \alpha\end{bmatrix}'\right)_0 = 0.$$

Multiplying this by $(g'^{\alpha e})_0$ and summing as to α leads to the first of (61) so that (61) is equivalent to

$$(62)\qquad \begin{cases} \left(\dfrac{\partial}{\partial x'_\sigma}\begin{bmatrix}\mu\nu\\ e\end{bmatrix}'\right)_0 + \left(\dfrac{\partial}{\partial x'_\nu}\begin{bmatrix}\mu\sigma\\ e\end{bmatrix}'\right)_0 + \left(\dfrac{\partial}{\partial x'_\mu}\begin{bmatrix}\nu\sigma\\ e\end{bmatrix}'\right)_0 = 0 \\ -\left(\dfrac{\partial}{\partial x'_\sigma}\begin{bmatrix}\mu\nu\\ e\end{bmatrix}'\right)_0 + \left(\dfrac{\partial}{\partial x'_\nu}\begin{bmatrix}\mu\sigma\\ e\end{bmatrix}'\right)_0 = (R'_{e\mu\nu\sigma})_0. \end{cases}$$

To the sum of these two equations add the second with μ and σ interchanged; thus

$$\left(\frac{\partial}{\partial x'_\nu}\begin{bmatrix}\mu\sigma\\ e\end{bmatrix}'\right)_0 = \frac{1}{3}[(R'_{e\mu\nu\sigma})_0 + (R'_{e\sigma\nu\mu})_0].$$

If we substitute this in (62) we are led by (50) to identities; hence this last is equivalent to (62). If to this last we add the equation obtained from it by interchanging σ and e we find, by (19),

$$\left(\frac{\partial^2 g_{\mu\nu}'}{\partial x_\rho' \partial x_\sigma'}\right)_0 = \frac{1}{3}(R'_{\rho\mu\nu\sigma} + R'_{\rho\nu\mu\sigma})_0$$

as the solution of (61).

Thus

$$(A)_{P_0} = A\left((g'_{\mu\nu})_0,\ 0,\ \tfrac{1}{3}(R'_{\rho\mu\nu\sigma} + R'_{\rho\nu\mu\sigma})_0\right).$$

But by (60)

$$(g'_{\mu\nu})_0 = (g_{\mu\nu})_{P_0},\ (R'_{\rho\mu\nu\sigma})_0 = (R_{\rho\mu\nu\sigma})_{P_0}\ ;$$

hence at P_0

$$A = A\left(g_{\mu\nu},\ 0,\ \tfrac{1}{3}(R_{\rho\mu\nu\sigma} + R_{\rho\nu\mu\sigma})\right). \tag{63}$$

Since P_0 is an arbitrary non-singular point (63) is true at all such points. We thus have the important theorem:†

Every fundamental tensor which does not contain derivatives of the $g_{\mu\nu}$ beyond the second order is an explicit function of only the $g_{\mu\nu}$ and the $R_{\rho\mu\nu\sigma}$; if this tensor is linear (with constant coefficients) in the second order derivatives of $g_{\mu\nu}$ it is linear (with constant coefficients) in the $R_{\rho\mu\nu\sigma}$.

This latter tensor has the form

$$\sum_{\rho,\mu,\nu,\sigma} M(\mu_1,\ \cdots,\ \mu_m\ ;\ \rho,\mu,\nu,\sigma)(R_{\rho\mu\nu\sigma} + R_{\rho\nu\mu\sigma}) + N(\mu_1,\ \cdots\ ,\ \mu_m), \tag{64}$$

where M and N are functions of the $g_{\mu\nu}$ alone.

We have at once from the theorem:

Every fundamental tensor which does not contain derivatives of $g_{\mu\nu}$ beyond the first order is an explicit function of only the $g_{\mu\nu}$.

† For a proof of this theorem independent of tensors see E. Cartan: *Journal de Mathématiques*, 1922.

CHAPTER V

FUNDAMENTAL TENSORS

§30. *Fundamental Non-Differential Tensors of a Linear Differential Form.*
Consider the single linear differential form

$$dr = \sum_{\mu} h_\mu dx_\mu. \tag{1}$$

By a fundamental non-differential tensor of (1) we shall mean a tensor whose components are explicit functions of only $h_1, \cdots, h_4$.† It is of interest to investigate how many and what types of such tensors exist. We shall confine our attention to tensors whose components are analytic functions of the h_μ, and hence expressible as power series in the h_μ.

A notable simplification in all of the proofs is attained by working in two dimensions instead of four. A circumspect examination of the proofs will show that the same method is effective in the four-dimensional case. We shall use the following notation:

$$\rho_{ij} \equiv \frac{\partial x_i}{\partial x_j'}$$

$$\sigma_{ij} \equiv \frac{\partial x_i'}{\partial x_j} \cdot$$

Hence the matrices ρ and σ are reciprocals.

THEOREM 1: *Every invariant matrix is a matrix of constants.*

Let $f(h_\mu)$ be a component of an invariant matrix. Then, by definition, for every (T)

$$f'(h_\mu) = f(h_\mu), \qquad \text{(for every } h_\mu). \tag{65}$$

We may write

$$f(h_\mu) = \sum_{i,j}^{0,\infty} a_{ij} h_1^i h_2^j,$$

the a_{ij} being constants. Then

$$f'(h_\mu) = \sum_{i,j}^{0,\infty} a_{ij} h_1'^i h_2'^j.$$

† In this section the convention that h_μ shall stand for $h_\mu^{(i)}$ has been suspended.

Now since (65) is true for every (T), it is true for a (T) such that $\rho_{12}=\rho_{21}=0$. Then by (3)

$$(66) \qquad h_1' = \rho_{11}h_1, \quad h_2' = \rho_{22}h_2.$$

Therefore by (65)

$$\sum_{i,j}^{0,\infty} a_{ij}\rho_{11}^i\rho_{22}^j h_1^i h_2^j = \sum_{i,j}^{0,\infty} a_{ij}h_1^i h_2^j .$$

Since this is true for every h_1, h_2, it follows that

$$a_{ij}\rho_{11}^i\rho_{22}^j = a_{ij}.$$

Since this is true for every $\rho_{11}\neq 0$, $\rho_{22}\neq 0$, we see that all the a_{ij} are zero except a_{00}. Hence $f(h_1, h_2)=a_{00}=$ constant. Thus every component of an invariant matrix is a constant.

THEOREM 2: *Every covariant tensor of valence one is a constant multiple of the tensor h.*

Let the components of the tensor be

$$A_1(h_1,h_2) = \sum_{i,j} a_{ij}h_1 h_2$$

$$A_2(h_1,h_2) = \sum_{i,j} b_{ij}h_1^i h_2^j .$$

By (3)

$$(67) \qquad \begin{aligned} A_1' &= \rho_{11}A_1 + \rho_{21}A_2 \\ A_2' &= \rho_{12}A_1 + \rho_{22}A_2. \end{aligned}$$

Now with $\rho_{12}=\rho_{21}=0$ we have

$$A_1' = \sum_{i,j} a_{ij}h_1'^i h_2'^j = \sum_{i,j} a_{ij}\rho_{11}^i\rho_{22}^j h_1^i h_2^j ,$$

$$\rho_{11}A_1 = \sum_{i,j} a_{ij}\rho_{11}h_1^i h_2^j ,$$

from which it follows that

$$a_{ij}\rho_{11}^i\rho_{22}^j = a_{ij}\rho_{11}.$$

Hence all a_{ij} are zero except $a_{10}\equiv a$. Similarly all b_{ij} are zero except $b_{01}\equiv b$. Hence $A_1=ah_1$; $A_2=bh_2$. It remains to show that $a=b$. To this end consider a general (T); then

$$A_1' = ah_1' = a\rho_{11}h_1 + a\rho_{21}h_2$$

$$\rho_{11}A_1 + \rho_{21}A_2 = a\rho_{11}h_1 + b\rho_{21}h_2.$$

By (67) these two quantities are equal, and therefore $a=b$.

THEOREM 3: *Every covariant tensor of valence two is a constant multiple of the tensor* $h_{(1)}\, h_{(1)}$.

This theorem is a special case of Theorem 4. We merely wish to point out in passing that the matrix $h_{(1)}h_{(1)}$ is singular, that is, its determinant is zero; hence it has no reciprocal.

THEOREM 4: *Every tensor of type* $\binom{0}{m}$ *is a constant multiple of the tensor* $h_{(1)} \overset{m}{\cdots} h_{(1)}$.

For a (T) with $\rho_{12}=\rho_{21}=0$ we obtain by the same method as in previous theorems

$$
\begin{aligned}
A_{11\ldots 11} &= a_{11\ldots 11}h_1^m \\
A_{11\ldots 12} &= a_{11\ldots 12}h_1^{m-1}h_2 \\
&\cdots\cdots\cdots \\
A_{21\ldots 11} &= a_{21\ldots 11}h_1^{m-1}h_2 \\
&\cdots\cdots\cdots \\
A_{22\ldots 21} &= a_{22\ldots 21}h_1h_2^{m-1} \\
&\cdots\cdots\cdots \\
A_{12\ldots 22} &= a_{12\ldots 22}h_1h_2^{m-1} \\
A_{22\ldots 22} &= a_{22\ldots 22}h_2^m.
\end{aligned}
$$

Now for a (T) with $\rho_{11}=\rho_{22}=0$ we find that a is symmetric with respect to the subscripts 1, 2; for example $a_{11\ldots 12}=a_{22\ldots 21}$. Let us now consider a general (T). Then

$$(68)\quad A'_{11\ldots 11} = a_{11\ldots 11}(\rho_{11}h_1 + \rho_{21}h_2)^m = a_{11\ldots 11}\sum_{k=0}^{m} C^m_k \rho_{11}^{k}\rho_{21}^{m-k} h_1^k h_2^{m-k}.$$

But by (7)

$$A'_{11\ldots 11} = \rho_{11}^{m} A_{11\ldots 11} + \rho_{11}^{m-1}\rho_{21}A_{11\ldots 12} + \cdots + \rho_{21}^{m}A_{22\ldots 22}.$$

The coefficient of $h_1^{m-1}h_2$ in the right member of the last equation is

$$\rho_{11}^{m-1}\rho_{21}(a_{11\ldots 12} + a_{11\ldots 21} + \cdots + a_{21\ldots 11}).$$

The coefficient of $h_1^{m-1}h_2$ in the right member of (68) is $\rho_{11}^{m-1}\rho_{21}\, m a_{11\ldots 11}$. Hence

$$(69)\qquad a_{11\ldots 12} + a_{11\ldots 21} + \cdots + a_{21\ldots 11} = m a_{11\ldots 11}.$$

Similarly, starting with $A'_{11\ldots 12}$ instead of $A'_{11\ldots 11}$, and again considering the coefficients of $h_1^{m-1}h_2$, we obtain

$$(1-m)a_{11\ldots 12} + a_{11\ldots 21} + \cdots + a_{21\ldots 11} = 0.$$

In the same way, starting with the remaining $(m-1)$ components A' which have only one subscript equal to 2, we obtain in all m homogeneous linear equations with constant coefficients in the m quantities $a_{(\text{one } 2)}$ whose determinant D is as follows: each term of the principal diagonal has the value $1-m$; all other terms have the value 1.

LEMMA. *The k-order determinant whose principal diagonal elements are* $1-m$, *all other elements being* 1, *has the value*

$$(-1)^{k-1}(k-m)m^{k-1}.$$

For the proof see Appendix I. For $k=m$ this quantity is zero; for $k=m-1$ it is distinct from zero.

Hence D is of rank $m-1$, so that the m equations in question have a unique solution for the $m-1$ ratios of the m quantities $a_{(\text{one } 2)}$. These equations are clearly satisfied when these m quantities are all equal. Hence this is the only solution.

Now starting with any one of the remaining $A' \ldots$, and again considering the coefficients of $h_1^{m-1}h_2$, we find that $a \ldots = a_{11 \ldots 11}$, where $A' \ldots$ and $a \ldots$ are any corresponding pair. Hence the tensor is equal to a constant multiple of $h_{(1)} \cdot \overset{m}{\cdots} h_{(1)}$.

THEOREM 5: *Every tensor of type* $\binom{n}{m}$, *where* $n > m$, *is zero.*

We select a (T) with $\rho_{12}=\rho_{21}=0$. Then

$$A'^{11\cdots 11}_{11\cdots 11} = \overset{m}{\rho_{11}}\overset{n}{\sigma_{11}} A^{11\cdots 11}_{11\cdots 11}.$$

Hence from (66)

$$a_{ij}\overset{i}{\rho_{11}}\overset{j}{\rho_{22}} = \overset{m}{\rho_{11}}\overset{n}{\sigma_{11}} a_{ij} = \frac{a_{ij}}{\rho_{11}^{n-m}},$$

since here $\sigma_{11} = 1/\rho_{11}$. Since $n-m>0$ it follows that all the a_{ij} are zero.

The same method shows that all the other components of the tensor are zero.

THEOREM 6: *Every tensor of type* $\binom{m}{m}$ *is an invariant, and hence a matrix of constants.*

Using a (T) for which $\rho_{12}=\rho_{21}=0$ we obtain

$$A'^{11\cdots 11}_{11\cdots 11} = \overset{m}{\rho_{11}}\overset{m}{\sigma_{11}} A^{11\cdots 11}_{11\cdots 11} = A^{11\cdots 11}_{11\cdots 11}.$$

Similarly we find that all the other components are either zero or invariant. Hence the tensor is invariant, and by Theorem 1 is a matrix of constants.

THEOREM 7: *Every matrix of type* $\binom{1}{1}$ *is a constant multiple of the identity matrix I of the second order.*

For by Theorem 6 a matrix of type $\binom{1}{1}$ is a matrix of constants $\binom{a\ b}{c\ d}$. Now for a (T) with $\rho_{12}=\rho_{21}=0$ we find $b=c=0$. Now by using a (T) with $\rho_{11}=\rho_{22}=0$ we find $a=d$. Here $A^{(1)}_{(1)}=aI$.

THEOREM 8: *Every tensor of type $\binom{2}{2}$ has the form*

$$\begin{cases} A^{11}_{11} = A^{22}_{22} = a+b \\ A^{12}_{12} = A^{21}_{21} = a \\ A^{21}_{12} = A^{12}_{21} = b \\ \text{other ten components zero.} \end{cases}$$

For a (T) with $\rho_{12}=\rho_{21}=0$ we find that ten of the components are zero, as stated in the theorem. Using a general (T) it now follows readily that the six remaining components have the form stated.

We may note that for $b=0$ this tensor reduces to aII.

There remains but one case to consider, namely, a tensor of type $\binom{m}{m+n}$, where $m>0$, $n>0$. This problem we have been unable to solve. We believe, however, that the following proposition is correct:

PROPOSITION: *Every tensor of type $\binom{m}{m+n}$ may be written as the product of a tensor of type $\binom{m}{m}$ and a tensor of type $\binom{0}{n}$, that is*

$$A^{(m)}_{(m+n)} = B^{(m)}_{(m)}C^{(0)}_{(n)} = K^{(m)}_{(m)}h_{(1)} \cdot \overset{n}{\smile} \cdot h_{(1)},$$

where $K^{(m)}_{(m)}$ is a matrix of constants.

In connection with this proposition we have been able to show only the following, using a (T) with $\rho_{12}=\rho_{21}=0$:

$$A^{(p\text{ ones},\ r\text{ twos})}_{(k\text{ ones},\ l\text{ twos})} = \begin{cases} 0 \text{ if } p>k \text{ or } r>l \\ a^{(p\text{ ones},\ r\text{ twos})}_{(k\text{ ones},\ l\text{ twos})}\ h_1^{k-p}\ \ h_2^{l-r}. \end{cases}$$

Since $k+l=m+n$, $p+r=m$, we see that $(k-p)+(l-r)=n$. Now using a (T) for which $\rho_{11}=\rho_{22}=0$ we find that $a^{\cdot\cdot}_{\cdot\cdot}$ is symmetric with respect to the indices 1, 2. To prove the proposition it would still be necessary to show that certain of the $a_{\cdot}$ are equal to each other.

THEOREM 9: *Every tensor of type $\binom{1}{2}$ is a constant multiple of the product of the tensor I and the tensor h.*

For it follows from what we have proved in connection with the above proposition that the components of the tensor $A^{(1)}_{(2)}$ are as follows:

$$
(70)\qquad
\begin{aligned}
\overset{1}{A}_{11} &= \overset{1}{a}_{11}h_1 = ah_1, & \overset{2}{A}_{22} &= ah_2\\
\overset{1}{A}_{12} &= \overset{1}{a}_{12}h_2 = bh_2, & \overset{2}{A}_{21} &= bh_1\\
\overset{1}{A}_{21} &= \overset{1}{a}_{21}h_2 = ch_2, & \overset{2}{A}_{12} &= ch_1\\
\overset{1}{A}_{22} &= 0, & \overset{2}{A}_{11} &= 0.
\end{aligned}
$$

Now let us use a general (T). We then have

$$
\begin{aligned}
\overset{\prime 1}{A}_{11} = \overset{2}{\rho}_{11}\sigma_{11}\overset{1}{A}_{11} + \rho_{11}\rho_{21}\sigma_{11}\overset{1}{A}_{12} + \rho_{11}\rho_{21}\sigma_{11}\overset{1}{A}_{21} + \rho_{11}\rho_{21}\sigma_{12}\overset{2}{A}_{12}\\
+ \rho_{21}\rho_{11}\sigma_{12}\overset{2}{A}_{21} + \overset{2}{\rho}_{21}\sigma_{12}\overset{2}{A}_{22}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
h_1' &= \rho_{11}h_1 + \rho_{21}h_2\\
h_2' &= \rho_{12}h_1 + \rho_{22}h_2.
\end{aligned}
$$

Hence, using (70)

$$
\begin{aligned}
a\rho_{11}h_1 + a\rho_{21}h_2 = a\frac{\overset{2}{\rho}_{11}\rho_{22}}{\det \rho}h_1 + b\frac{\rho_{11}\rho_{21}\rho_{22}}{\det \rho}h_2 + c\frac{\rho_{11}\rho_{21}\rho_{22}}{\det \rho}h_2\\
- c\frac{\rho_{11}\rho_{12}\rho_{21}}{\det \rho}h_1 - b\frac{\rho_{11}\rho_{12}\rho_{21}}{\det \rho}h_1 - a\frac{\rho_{12}\overset{2}{\rho}_{21}}{\det \rho}h_2.
\end{aligned}
$$

Equating coefficients of h_1 and h_2 we obtain

$$c+b=a.$$

Proceeding similarly with $\overset{\prime 1}{A}_{12}$ we obtain

$$b=a.$$

Therefore $c=0$, and the matrix A has the components

$$
\begin{aligned}
\overset{1}{A}_{11} &= ah_1 & \overset{1}{A}_{21} &= 0\\
\overset{1}{A}_{12} &= ah_2 & \overset{1}{A}_{22} &= 0\\
\overset{2}{A}_{21} &= ah_1 & \overset{2}{A}_{12} &= 0\\
\overset{2}{A}_{22} &= ah_2 & \overset{2}{A}_{11} &= 0.
\end{aligned}
$$

This matrix may be written aIh.

§31 *Fundamental Non-Differential Tensors of a Quadratic Differential Form.*

We shall now treat fundamental non-differential tensors of the quadratic differential form

$$(41)\qquad ds^2 = \sum_{\mu,\nu} g_{\mu\nu}dx_\mu dx_\nu.$$

THEOREM 1: *Every invariant tensor is a matrix of constants.*

The proof is entirely analogous to that of Theorem 1 of §30.

THEOREM 2: *Every tensor of type* $\binom{0}{2m+1}$ *is zero.*

We use a (T) with $\rho_{12}=\rho_{21}=0$. Then

$$A'_{11\cdots 11} = \rho_{11}^{2m+1} A_{11\cdots 11}$$

$$g'_{\mu\nu} = \rho_{\mu\mu}\rho_{\nu\nu}g_{\mu\nu}.$$

Putting

$$A_{11\cdots 11} = \sum_{i,j,k}^{1,\infty} a_{ijk}g_{11}^{i}g_{12}^{j}g_{22}^{k}$$

we have

$$a_{ijk}\rho_{11}^{2i+j}\rho_{22}^{j+2k} = a_{ijk}\rho_{11}^{2m+1}.$$

Hence $a_{ijk}=0$ for all i, j, k, and therefore $A_{11\cdots 11}=0$. In exactly the same way it turns out that all the other components of the tensor are zero.

THEOREM 3: *Every tensor of odd valence is zero.*

Consider the tensor $A\binom{n}{m}$, where $m+n$ is odd. By n successive associations of this tensor with $g_{(2)}$ we obtain a tensor of type $\binom{0}{m+n}$, which is equal to zero by Theorem 2. Hence it follows from §11 that the tensor $A\binom{n}{m}$ is zero.

THEOREM 4: *Every tensor of type* $\binom{0}{2}$ *is a constant multiple of the tensor* $g_{(2)}$.

This proof is entirely analogous to that of Theorem 2 of §30.

THEOREM 5: *Every tensor of type* $\binom{1}{1}$ *is a constant multiple of the identity matrix* I.

For a (T) with $\rho_{12}=\rho_{21}=0$ we find at once that

$$A_1^1 = a, \qquad A_1^2 = 0$$
$$A_2^2 = b, \qquad A_2^1 = 0.$$

Now for a (T) with $\rho_{11}=\rho_{22}=0$ we find that $a=b$. Hence $A\binom{1}{1}=aI$.

THEOREM 6: *Every tensor of type* $\binom{2}{0}$ *is a constant multiple of* $g^{(2)}$.

Let $A^{(2)}$ be a tensor of type $\binom{2}{0}$. Then

$$\sum_{\nu} A^{\mu\nu}g_{\nu\sigma} = aI(\mu,\sigma)$$

by Theorem 5. Hence

$$\sum_{\nu,\sigma} A^{\mu\nu} g_{\nu\sigma} g^{\sigma\tau} = a \sum_{\sigma} g^{\sigma\tau} I(\mu,\sigma) = a g^{\mu\tau}.$$

But the left member of this equation equals $A^{\mu\tau}$. Therefore $A^{(2)} = a g^{(2)}$.

THEOREM 7: *There exist exactly three linearly independent tensors of type* $\binom{0}{4}$, *namely* $g_{(2)} g_{(2)}$ *with the indices in all possible orders:*

$$g_{\mu\nu} g_{\rho\sigma}, \quad g_{\mu\rho} g_{\nu\sigma}, \quad g_{\mu\sigma} g_{\nu\rho}.$$

For a (T) with $\rho_{12} = \rho_{21} = 0$ we have

$$A'_{1111} = \rho_{11}^{4} A_{1111}.$$

Hence

$$a_{ijk} \rho_{11}^{2i+j} \rho_{22}^{j+2k} = \rho_{11}^{4} a_{ijk}.$$

Thus

$$A_{1111} = a g_{11}^{2} .$$

In the same way, for the other components we find

$$(71) \qquad \begin{aligned} A_{1111} &= a g_{11}^{2}, & A_{2222} &= \bar{a} g_{22}^{2} \\ A_{1112} &= b g_{11} g_{12}, & A_{2221} &= \bar{b} g_{12} g_{22} \\ A_{1121} &= c g_{11} g_{12}, & A_{2212} &= \bar{c} g_{12} g_{22} \\ A_{1211} &= d g_{11} g_{12}, & A_{2122} &= \bar{d} g_{12} g_{22} \\ A_{2111} &= e g_{11} g_{12}, & A_{1222} &= \bar{e} g_{12} g_{22} \\ A_{1122} &= \alpha g_{11} g_{22} + \delta g_{12}^{2}, & A_{2211} &= \bar{\alpha} g_{11} g_{22} + \bar{\delta} g_{12}^{2} \\ A_{1212} &= \beta g_{11} g_{22} + \epsilon g_{12}^{2}, & A_{2121} &= \bar{\beta} g_{11} g_{22} + \bar{\epsilon} g_{12}^{2} \\ A_{2112} &= \gamma g_{11} g_{22} + \eta g_{12}^{2}, & A_{1221} &= \bar{\gamma} g_{11} g_{22} + \bar{\eta} g_{12}^{2}, \end{aligned}$$

where a, $\bar{a}$, b, $\bar{b}$, α, $\bar{\alpha}$ etc. are constants. Using now a (T) with $\rho_{11} = \rho_{22} = 0$ we readily find that

$$(72) \qquad \begin{aligned} a &= \bar{a} \\ b &= \bar{b} \\ \alpha &= \bar{\alpha} \\ &\text{etc..} \end{aligned}$$

Now we employ a general (T). Then

$$\begin{aligned} A'_{1112} = {} & a\rho_{11}^{3}\rho_{12}g_{11}^{2} + b\rho_{11}^{3}\rho_{22}g_{11}g_{12} + (c+d+e)\rho_{11}^{2}\rho_{21}\rho_{12}g_{11}g_{12} \\ & + (\alpha+\beta+\gamma)\rho_{11}^{2}\rho_{21}\rho_{22}g_{11}g_{22} + (\delta+\epsilon+\eta)\rho_{11}^{2}\rho_{21}\rho_{22}g_{12}^{2} \\ & + a\rho_{21}^{3}\rho_{22}g_{22}^{2} + b\rho_{21}^{3}\rho_{12}g_{12}g_{22} + (c+d+e)\rho_{11}\rho_{21}^{2}\rho_{22}g_{12}g_{22} \\ & + (\alpha+\beta+\gamma)\rho_{11}\rho_{21}^{2}\rho_{12}g_{11}g_{22} + (\delta+\epsilon+\eta)\rho_{11}\rho_{21}^{2}\rho_{12}g_{12}^{2}. \end{aligned}$$

On the other hand

$$A'_{1112} = b(\rho_{11}^{2}g_{11} + 2\rho_{11}\rho_{21}g_{12} + \rho_{21}^{2}g_{22})(\rho_{11}\rho_{12}g_{11} + \rho_{11}\rho_{22}g_{12} + \rho_{21}\rho_{12}g_{21} + \rho_{21}\rho_{22}g_{22}).$$

Equating coefficients for the terms $\rho_{11}^{3}\rho_{12}g_{11}^{2}$, $\rho_{11}\rho_{21}^{2}\rho_{12}g_{11}g_{22}$, $\rho_{11}\rho_{21}^{2}\rho_{12}g_{12}^{2}$, we obtain

$$\begin{aligned} & b = a \\ (73) \qquad & \alpha+\beta+\gamma = b \\ & \delta+\epsilon+\eta = 2b. \end{aligned}$$

Similarly from A'_{1221}, A'_{1211}, A'_{2111}, we find

$$(74) \qquad c = d = e = a.$$

Now from A'_{1122} and A'_{1212} we obtain by the same method

$$\begin{aligned} & \alpha+\delta = a \\ (75) \qquad & \beta+\epsilon = a \\ & \beta+\gamma = \delta \\ & \alpha+\gamma = \epsilon. \end{aligned}$$

Now by (72), (73), (74), (75), we may write (71) as

$$\begin{aligned} & A_{1111} = (\alpha+\beta+\gamma)g_{11}^{2} \\ & A_{1112} = A_{1121} = A_{1211} = A_{2111} = (\alpha+\beta+\gamma)g_{11}g_{12} \\ & A_{2222} = (\alpha+\beta+\gamma)g_{22}^{2} \\ & A_{2221} = A_{2212} = A_{2122} = A_{1222} = (\alpha+\beta+\gamma)g_{12}g_{22} \\ & A_{1122} = A_{2211} = \alpha g_{11}g_{22} + (\beta+\gamma)g_{12}^{2} \\ & A_{1212} = A_{2121} = \beta g_{11}g_{22} + (\alpha+\gamma)g_{12}^{2} \\ & A_{2112} = A_{1221} = \gamma g_{11}g_{22} + (\alpha+\beta)g_{12}^{2}. \end{aligned}$$

This result may be represented by the following formula:

$$A_{\mu\nu\rho\sigma} = \alpha g_{\mu\nu}g_{\rho\sigma} + \beta g_{\mu\rho}g_{\nu\sigma} + \gamma g_{\mu\sigma}g_{\nu\rho}, \tag{76}$$

from which the theorem follows directly.

THEOREM 8: *There exist exactly three linearly independent tensors of type* $\binom{4}{0}$, *namely* $g^{(2)}g^{(2)}$, *with the indices in all possible orders.*

The proof is completely analogous to that of Theorem 7.

THEOREM 9: *There exist exactly three linearly independent tensors of type* $\binom{2}{2}$, *namely those whose components are*

$$g_{\mu\nu}g^{\rho\sigma},\ I(\mu,\rho)I(\nu,\sigma),\ I(\nu,\rho)I(\mu,\sigma).$$

In proof consider a tensor $A^{\rho\sigma}_{\mu\nu}$. Then by §11

$$\sum_{\rho,\sigma} A^{\rho\sigma}_{\mu\nu}g_{\rho\alpha}g_{\sigma\beta} = A_{\mu\nu\alpha\beta}.$$

Hence

$$\sum_{\alpha,\beta,\rho,\sigma} A^{\rho\sigma}_{\mu\nu}g_{\rho\alpha}g_{\sigma\beta}g^{\alpha\gamma}g^{\beta\delta} = \sum_{\alpha,\beta} A_{\mu\nu\alpha\beta}g^{\alpha\gamma}g^{\beta\delta}.$$

The left member reduces to $A^{\gamma\delta}_{\mu\nu}$. Therefore

$$A^{\rho\sigma}_{\mu\nu} = \sum_{\alpha,\beta} g^{\alpha\rho}g^{\beta\sigma}A_{\mu\nu\alpha\beta}$$

Using (76) we find that

$$A^{\rho\sigma}_{\mu\nu} = \alpha g_{\mu\nu}g^{\rho\sigma} + \beta I(\mu,\rho)I(\nu,\sigma) + \gamma I(\nu,\rho)I(\mu,\sigma). \tag{77}$$

THEOREM 10: *The number of linearly independent tensors of type* $\binom{0}{2m}$ *is*

$$N(m) = \prod_{k=1}^{m} (2m - 2k + 1) = 1\cdot 3\cdot 5 \cdots (2m-1).$$

A complete set of linearly independent ones is given by m *successive multiplications of* $g_{(2)}$, *with all possible orders of indices.*†

For the proof see Appendix II, Theorem II.

THEOREM 11: *A complete set of linearly independent tensors of type* $\binom{2m}{0}$ *is given by* m *successive multiplications of* $g^{(2)}$, *with all possible orders of indices. The number of such tensors is* $N(m)$.

The proof is analogous to that of Theorem 10.

† We find that this theorem was given by P. Franklin: Phil. Mag., vol. 45, 1923, p. 998.

THEOREM 12: *A complete set of linearly independent tensors of type* $\binom{m}{m}$ *is obtained by repeated outer multiplication of* $g_{(2)}$, $g^{(2)}$, *and* I. *The number of such linearly independent tensors is* $N(m)$.

This follows from Theorem 10 in the same way as Theorem 9 follows from Theorem 7.

THEOREM 13: *A complete set of linearly independent tensors of type* $\binom{n}{2m-n}$ *is obtained by repeated outer multiplications of* $g_{(2)}$, $g^{(2)}$, *and* I. *The number of such linearly independent tensors is* $N(m)$.

§32. *Fundamental Differential Tensors of a Quadratic Differential Form.*

We shall consider only fundamental tensors which do not contain derivatives of the $g_{\mu\nu}$ beyond the second order, and which are linear in the second order derivatives. By §29 every covariant tensor of this kind may be written

(78) $$\overset{(0)}{A}_{(m)} = \sum_{\rho,\mu,\nu,\sigma} M(\tau_1, \cdots, \tau_m ; \rho, \mu, \nu, \sigma)(R_{\rho\mu\nu\sigma} + R_{\rho\nu\mu\sigma}) + N(\tau_1 \cdots \tau_m)$$

where M and N are functions of the $g_{\mu\nu}$ alone.

We shall now show that (78) may be written as

(79) $$\overset{(0)}{A}_{(m)} = \sum_{\rho,\mu,\nu,\sigma} P(\tau_1, \cdots, \tau_m; \rho, \mu, \nu, \sigma)(R_{\rho\mu\nu\sigma} + R_{\rho\nu\mu\sigma}) + N(\tau_1, \cdots, \tau_m);$$

where $P(\tau_1, \cdots, \tau_m; \rho, \mu, \nu, \sigma)$ has the same symmetry relations in ρ, μ, ν, σ as does $R_{\rho\mu\nu\sigma}$ by (50). To prove this let us proceed as follows. There are exactly 20 linearly independent $R_{\rho\mu\nu\sigma}$, a complete set of which is given in §26. We here give a table of the $R_{\rho\mu\nu\sigma}$, using the notation $\rho\mu\nu\sigma$ for $R_{\rho\mu\nu\sigma}$:

(80)
$$\begin{aligned}
&1212 = -2112 = -1221 = 2121\\
&1213 = -2113 = -1231 = 2131 = 1312 = -3112 = -1321 = 3121\\
&1214 = -2114 = -1241 = 2141 = 1412 = -4112 = -1421 = 4121\\
&1223 = -2123 = -1232 = 2132 = 2312 = -3212 = -2321 = 3221\\
&1224 = -2124 = -1242 = 2142 = 2412 = -4212 = -2421 = 4221\\
&1234 = -2134 = -1243 = 2143 = 3412 = -4312 = -3421 = 4321\\
&1313 = -3113 = -1331 = 3131\\
&1314 = -3114 = -1341 = 3141 = 1413 = -4113 = -1431 = 4131\\
&1323 = -3123 = -1332 = 3132 = 2313 = -3213 = -2331 = 3231
\end{aligned}$$

$$
(80)\quad
\begin{aligned}
&1324 = -3124 = -1342 = 3142 = 2413 = -4213 = -2431 = 4231\\
&1334 = -3134 = -1343 = 3143 = 3413 = -4313 = -3431 = 4331\\
&1414 = -4114 = -1441 = 4141\\
&1424 = -4124 = -1442 = 4142 = 2414 = -4214 = -2441 = 4241\\
&1434 = -4134 = -1443 = 4143 = 3414 = -4314 = -3441 = 4341\\
&2323 = -3223 = -2332 = 3232\\
&2324 = -3224 = -2342 = 3242 = 2423 = -4223 = -2432 = 4232\\
&2334 = -3234 = -2343 = 3243 = 3423 = -4323 = -3432 = 4332\\
&2424 = -4224 = -2442 = 4242\\
&2434 = -4234 = -2443 = 4243 = 3424 = -4324 = -3442 = 4342\\
&3434 = -4334 = -3443 = 4343\,.
\end{aligned}
$$

Eight other components are given in terms of these by

$$
(81)\quad
\begin{aligned}
&1423 = -4123 = -1432 = 4132 = 2314 = -3214 = -2341 = 3241\\
&\quad = -1234 + 1324.
\end{aligned}
$$

The remaining 112 components are all zero.

We shall also abbreviate by writing $P(\rho, \mu, \nu, \sigma)$ for $P(\tau_1, \cdots, \tau_m; \rho, \mu, \nu, \sigma)$, and $M(\rho, \mu, \nu, \sigma)$ for $M(\tau_1, \cdots, \tau_m; \rho, \mu, \nu, \sigma)$. Now let us choose

$$
(82)\quad
\begin{aligned}
P(1,2,1,2) &= \frac{1}{6}[M(1,2,1,2) - 2M(2,1,1,2) - 2M(1,2,2,1)\\
&\qquad + M(2,1,2,1)]\\
P(1,2,1,3) &= \frac{1}{12}[M(1,2,1,3) - 2M(2,1,1,3) - 2M(1,2,3,1)\\
&\qquad + M(2,1,3,1) + M(1,3,1,2) - 2M(3,1,1,2)\\
&\qquad - 2M(1,3,2,1) + M(3,1,2,1)]\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
P(3,4,3,4) &= \frac{1}{6}[M(3,4,3,4) - 2M(4,3,3,4) - 2M(3,4,4,3)\\
&\qquad + M(4,3,4,3)].
\end{aligned}
$$

The analogy between this display and (80) is direct. The remaining 236 components of P shall be chosen analogously to (50) as follows:

$$
(83)\quad
\begin{cases}
P(\rho,\mu,\nu,\sigma) = -P(\rho,\mu,\sigma,\nu) = P(\nu,\sigma,\rho,\mu)\\
P(\rho,\mu,\nu,\sigma) + P(\rho,\nu,\sigma,\mu) + P(\rho,\sigma,\mu,\nu) = 0.
\end{cases}
$$

Thus for each set of indices $(\tau_1, \cdots, \tau_m)$ the 256 components $P(\rho, \mu, \nu, \sigma)$

are expressible in terms of the 20 given in (82), in the same way as the 256 components $R_{\rho\mu\nu\sigma}$ are expressible in terms of the corresponding 20 given in §26. Thus $P(\rho, \mu, \nu, \sigma)$ has exactly the same symmetry relations in ρ, μ, ν, σ as does $R_{\rho\mu\nu\sigma}$.

With this choice of P it follows that

$$(84)\qquad \sum_{\rho,\mu,\nu,\sigma} P(\tau_1, \cdots, \tau_m; \rho,\mu,\nu,\sigma)(R_{\rho\mu\nu\sigma} + R_{\rho\nu\mu\sigma}) = \sum_{\rho,\mu,\nu,\sigma} M(\tau_1, \cdots, \tau_m; \rho,\mu,\nu,\sigma)(R_{\rho\mu\nu\sigma} + R_{\rho\nu\mu\sigma}).$$

For by (83), (80), and (50)

$$P(1,2,1,2)R_{1212} + P(1,2,1,2)R_{1122} + P(2,1,1,2)R_{2112} + P(2,1,1,2)R_{2112} + P(1,2,2,1)R_{1221} + P(1,2,2,1)R_{1221} + P(2,1,2,1)R_{2121} + P(2,1,2,1)R_{2211} = 6P(1,2,1,2)R_{1212},$$

and

$$M(1,2,1,2)R_{1212} + M(1,2,1,2)R_{1122} + M(2,1,1,2)R_{2112} + M(2,1,1,2)R_{2112} + M(1,2,2,1)R_{1221} + M(1,2,2,1)R_{1221} + M(2,1,2,1)R_{2121} + M(2\ 1,2,1)R_{2211}$$
$$= [M(1,2,1,2) - 2M(2,1,1,2) - 2M(1,2,2,1) + M(2,1,2,1)]R_{1212},$$

which by (82) is equal to $6P(1, 2, 1, 2)R_{1212}$. There will be exactly similar relations for each line of the display (80), and also for (81). Adding these 21 equations we obtain (84), and consequently we may replace (78) by (79).

Now according to (5)

$$(85)\qquad \sum_{\rho,\mu,\nu,\sigma} P'(\tau_1, \cdots, \tau_m; \rho,\mu,\nu,\sigma)(R'_{\rho\mu\nu\sigma} + R'_{\rho\nu\mu\sigma}) + N'(\tau_1, \cdots, \tau_m) = \sum_{\epsilon_1, \cdots, \epsilon_m} \frac{\partial x_{\epsilon_1}}{\partial x'_{\tau_1}} \cdots \frac{\partial x_{\epsilon_m}}{\partial x'_{\tau_m}} \left[\sum_{\alpha,\beta,\gamma,\delta} P(\epsilon_1, \cdots, \epsilon_m; \alpha,\beta,\gamma,\delta)(R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\beta\delta}) + N(\epsilon_1, \cdots, \epsilon_m) \right].$$

This equation is an identity in the $g_{\mu\nu}$ and their derivatives. By (8)

$$R_{\alpha\beta\gamma\delta} = \sum_{\rho,\mu,\nu,\sigma} \frac{\partial x'_\rho}{\partial x_\alpha} \frac{\partial x'_\mu}{\partial x_\beta} \frac{\partial x'_\nu}{\partial x_\gamma} \frac{\partial x'_\sigma}{\partial x_\delta} R'_{\rho\mu\nu\sigma}.$$

Substitute this in (85). Now since by (49) every term of $R'_{\rho\mu\nu\sigma}$ contains a derivative of $g'_{\mu\nu}$, the identity (85) leads to the two identities

$$(86)\qquad \sum_{\rho,\mu,\nu,\sigma} (R'_{\rho\mu\nu\sigma} + R'_{\rho\nu\mu\sigma}) \left[P'(\tau_1, \cdots, \tau_m; \rho,\mu,\nu,\sigma) - \sum_{\substack{\epsilon_1, \cdots, \epsilon_m \\ \alpha,\beta,\gamma,\delta}} \frac{\partial x_{\epsilon_1}}{\partial x'_{\tau_1}} \cdots \frac{\partial x_{\epsilon_m}}{\partial x'_{\tau_m}} \cdot \frac{\partial x'_\rho}{\partial x_\alpha} \frac{\partial x'_\mu}{\partial x_\beta} \frac{\partial x'_\nu}{\partial x_\gamma} \frac{\partial x'_\sigma}{\partial x_\delta} P(\epsilon_1, \cdots, \epsilon_m; \alpha,\beta,\gamma,\delta) \right] = 0$$

and

$$(87)\quad N'(\tau_1, \cdots, \tau_m) = \sum_{\epsilon_1, \cdots, \epsilon_m} \frac{\partial x_{\epsilon_1}}{\partial x_{\tau_1}'} \cdots \frac{\partial x_{\epsilon_m}}{\partial x_{\tau_m}'} N(\epsilon_1, \cdots, \epsilon_m).$$

Let us write (86) as

$$(88)\qquad \sum_{\rho,\mu,\nu,\sigma} (R_{\rho\mu\nu\sigma}' + R_{\rho\nu\mu\sigma}')\Lambda(\tau_1, \cdots, \tau_m; \rho,\mu,\nu,\sigma) = 0.$$

This equation is an identity in the $g_{\mu\nu}$ and their derivatives; consequently the coefficients of the different terms of (88) are separately zero.

Let us abbreviate by writing $\Lambda(\rho, \mu, \nu, \sigma)$ for $\Lambda(\tau_1, \cdots, \tau_m; \rho, \mu, \nu, \sigma)$. From the definition by (86) it follows readily that $\Lambda(\rho, \mu, \nu, \sigma)$ satisfies the same symmetry relations (50) as does $P(\rho, \mu, \nu, \sigma)$. By (49) the coefficient of $\partial^2 g_{12}'/\partial x_1' \partial x_2'$ in (88) is obtained, by the first line of (80), from

$$(R_{1212}' + R_{1122}')\Lambda(1,2,1,2) + (R_{2112}' + R_{2112}')\Lambda(2,1,1,2)$$

$$+ (R_{1221}' + R_{1221}')\Lambda(1,2,2,1) + (R_{2121}' + R_{2211}')\Lambda(2,1,2,1) = 6R_{1212}'\Lambda(1,2,1,2);$$

hence the coefficient is $6\Lambda(1, 2, 1, 2)$, and hence $\Lambda(1, 2, 1, 2) = 0$. The coefficient of $\partial^2 g_{13}'/\partial x_1' \partial x_2'$ is obtained similarly, by the second line of (80), from $12R_{1213}'\Lambda(1, 2, 1, 3)$, and is $6\Lambda(1, 2, 1, 3)$; hence $\Lambda(1, 2, 1, 3) = 0$. Similarly by (80) all the $\Lambda(\rho, \mu, \nu, \sigma)$ are zero except the 24 of the sixth and tenth lines and of (81); we now show that these are also zero.

The coefficients of $\partial^2 g_{12}'/\partial x_3' \partial x_4'$ and of $\partial^2 g_{13}'/\partial x_2' \partial x_4'$ are obtained from

$$\Lambda(1,2,3,4)(8R_{1234}' + 4R_{1324}' - 4R_{1423}')$$

$$+ \Lambda(1,3,2,4)(8R_{1324}' + 4R_{1234}' + 4R_{1423}')$$

$$+ \Lambda(1,4,2,3)(8R_{1423}' - 4R_{1234}' + 4R_{1324}').$$

Since $\Lambda(\rho, \mu, \nu, \sigma)$ and $R(\rho, \mu, \nu, \sigma)$ both satisfy the third symmetry relation of (50) this becomes

$$12(2R_{1234}' - R_{1324}')\Lambda(1,2,3,4) - 12(R_{1234}' - 2R_{1324}')\Lambda(1,3,2,4).$$

Thus the coefficients in question are respectively

$$6\Lambda(1,2,3,4) - 12\Lambda(1,3,2,4) \quad \text{and} \quad - 12\Lambda(1,2,3,4) + 6\Lambda(1,3,2,4).$$

Since these must be zero we see that $\Lambda(1, 2, 3, 4) = \Lambda(1, 3, 2, 4) = 0$; hence $\Lambda(1, 4, 2, 3) = 0$, and every component of Λ is zero. Therefore by the

definition of Λ it follows that P is a tensor of the type $\binom{4}{m}$, contravariant in ρ, μ, ν, σ.† By (87) N is a tensor of type $\binom{0}{m}$. By Theorem 3 of §31 either $P = N = 0$ or m is even.

We have thus proved the fundamental theorem:

Every covariant fundamental tensor which does not contain derivatives of the $g_{\mu\nu}$ beyond the second order, and which is linear in the second order derivatives, may be expressed in the form

$$\sum_{\rho,\mu,\nu,\sigma} P^{\rho\mu\nu\sigma}_{\tau_1,\cdots,\tau_{2m}}(R_{\rho\mu\nu\sigma}+R_{\rho\nu\mu\sigma}) + N_{\tau_1,\cdots,\tau_{2m}}, \tag{90}$$

where the tensors P and N are functions of the $g_{\mu\nu}$ alone and where P and R satisfy the same symmetry relations in ρ, μ, ν, σ.

§33. *Fundamental Differential Tensors of Odd Valence.*

By Theorem 3, §31 we see that there exists no non-zero fundamental non-differential tensor of odd valence. It follows from §32 that every non-zero fundamental differential tensor linear in the second order derivatives of the $g_{\mu\nu}$, and not containing any higher derivatives, is of even valence. Thus we have not yet found that there exists any fundamental tensor, differential or non-differential, of odd valence, except zero. But we can readily show that there do exist fundamental tensors of odd valence involving the third derivatives of $g_{\mu\nu}$; for we get such tensors by covariant differentiation of G, $G_{\mu\nu}$, and $R_{\rho\mu\nu\sigma}$. Thus

$$(91)\quad \left\{ \begin{array}{c} \dfrac{\partial G}{\partial x_\sigma} \\[2ex] \dfrac{\mathfrak{D} G_{\mu\nu}}{\mathfrak{D} x_\sigma} \\[2ex] \dfrac{\mathfrak{D} R_{\rho\mu\nu\sigma}}{\mathfrak{D} x_\tau} \end{array} \right.$$

are fundamental tensors of types $\binom{0}{1}$, $\binom{0}{3}$, and $\binom{0}{5}$ respectively, each involving third derivatives of the $g_{\mu\nu}$.

† Since 112 of the components of R are identically zero, the corresponding components of M in (78) may have any values whatever, so that M need not be a tensor.

CHAPTER VI

THE LAWS OF GRAVITATION

§34. *Postulates. Laws of Nature.*

We shall investigate the problem of possible laws of gravitation in free space† under the following set of postulates.

POSTULATE I. *The quadratic differential form*

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu} dx_\mu dx_\nu, \ det g_{\mu\nu} \neq 0, \tag{41}$$

defines the geometry of the physical universe.

As we have stated in §18, in the theory of spaces as developed by Gauss and Riemann, the properties of a space are equivalent to the properties of a symmetric quadratic differential form.

POSTULATE II. *The Fundamental Postulate of Relativity. Every law of nature is covariant.*

By this is meant that every law of nature consists of a set of covariant equations (§6). This postulate is the very essence of the Theory of Relativity, without which the theory ceases to exist.

POSTULATE III. *The Law of Gravitation is a set of differential equations in the $g_{\mu\nu}$ alone.*

PROPOSITION I. *Every set of covariant equations is equivalent to a set of tensor equations.*

As stated at the end of §6, we do not know that this is true. But it seems to be true, and we shall assume that it is until either proved or disproved; it must be stated that until such a time *the present development of the theory is incomplete.*

POSTULATE IV. *The Law of Gravitation is a set of differential equations not involving derivatives of $g_{\mu\nu}$ beyond the second order.*.

This postulate does not seem to be intrinsically essential; it is made, as is V, by analogy to classical dynamics.

POSTULATE V. *The Law of Gravitation is linear in the second order derivatives of the $g_{\mu\nu}$.*

This postulate also is not essential to a theory of Relativity.

† By free-space is meant all points of space free of matter or energy.

§35. *The Law of Gravitation.*

Assuming the correctness of the Proposition of §34, it follows from Postulate II that *every law of nature may be expressed by a set of tensor equations.* As the first law of nature we shall consider the Law of Gravitation, that is, the law which determines the $g_{\mu\nu}$ of (41) in terms of $x_1, \cdots, x_4$. From what we have said this law must be a set of tensor equations

$$A_i = 0 \qquad (i = 1, \cdots, m), \tag{92}$$

where each A_i is a tensor, a function of only $g_{\mu\nu}$, dx_μ, and the derivatives of the $g_{\mu\nu}$; for as we saw at the end of §3, the x_i cannot enter explicitly. Since the equations (92) are to determine the $g_{\mu\nu}$ in terms of the x_i alone, it follows† that the A_i are independent of the dx_i. Consequently the law of gravitation is a set of fundamental tensor equations. It now follows from Postulates IV and V, in view of the first theorem of §29, that the law of gravitation is a set of equations of the form of (64). Since by §11 the same law is obtained by equating to zero any one of a class of associated tensors, we may restrict ourselves to covariant tensors.‡ Consequently *the law of gravitation is a set of equations of the form of* (90).

We shall in the next section proceed to study the equation (90), thus obtaining all possible forms of the law of gravitation.

§36. *The Form of the Law of Gravitation.*

We shall consider for various values of m, the equation

$$\sum_{\rho,\mu,\nu,\sigma} P^{\rho\mu\nu\sigma}_{\tau_1,\cdots,\tau_{2m}}(R_{\rho\mu\nu\sigma} + R_{\rho\nu\mu\sigma}) + N_{\tau_1,\cdots,\tau_{2m}} \doteq 0, \tag{90}$$

where P and N are functions of the $g_{\mu\nu}$ alone, and where P and R satisfy the same symmetry relations, (50), in ρ, μ, ν, σ.

By a solution of (90) shall be meant a set of $g_{\mu\nu}$ with det $g_{\mu\nu} \neq 0$, satisfying (90).

CASE 1. $m=0$.

By Theorem 8 of §31 we have

$$P^{\rho\mu\nu\sigma} = \alpha g^{\rho\mu}g^{\nu\sigma} + \beta g^{\rho\nu}g^{\mu\sigma} + \gamma g^{\rho\sigma}g^{\mu\nu},$$

where α, β, γ are constants. Then by §27

$$\sum_{\rho,\mu,\nu,\sigma} P^{\rho\mu\nu\sigma}(R_{\rho\mu\nu\sigma} + R_{\rho\nu\mu\sigma}) = (-\alpha-\beta+2\gamma)G = aG.$$

† This argument is only intuitive; Postulate III permits us to avoid this difficulty.

‡ It was for this reason that we considered only covariant tensors in §32; as a matter of fact the method of that section suffices for mixed tensors as well.

By Theorem 1 of §31 we have $N=b$, a constant. Hence (90) becomes

$$aG+b=0. \tag{93}$$

If $a=0$, then the equation becomes merely $b=0$, which does not involve the $g_{\mu\nu}$. Therefore we need consider only the case $a\neq 0$, whence

$$G+c=0. \tag{94}$$

CASE 2. $m=1$.

By §31

$$\begin{aligned} P^{\rho\mu\nu\sigma}_{\tau_1\tau_2} = {} & \alpha_1 g^{\rho\mu} I^{\nu}_{\tau_1} I^{\sigma}_{\tau_2} + \alpha_2 g^{\rho\nu} I^{\mu}_{\tau_1} I^{\sigma}_{\tau_2} + \alpha_3 g^{\rho\sigma} I^{\mu}_{\tau_1} I^{\nu}_{\tau_2} \\ & + \alpha_4 g^{\mu\nu} I^{\rho}_{\tau_1} I^{\sigma}_{\tau_2} + \alpha_5 g^{\mu\sigma} I^{\rho}_{\tau_1} I^{\nu}_{\tau_2} + \alpha_6 g^{\nu\sigma} I^{\rho}_{\tau_1} I^{\mu}_{\tau_2} \\ & + \beta_1 g^{\rho\mu} I^{\sigma}_{\tau_1} I^{\nu}_{\tau_2} + \cdots + \beta_6 g^{\nu\sigma} I^{\mu}_{\tau_1} I^{\rho}_{\tau_2} \\ & + g_{\tau_1\tau_2}\left[\alpha_7 g^{\rho\mu} g^{\nu\sigma} + \alpha_8 g^{\rho\nu} g^{\mu\sigma} + \alpha_9 g^{\rho\sigma} g^{\mu\nu}\right]. \end{aligned}$$

Hence

$$\sum_{\rho,\mu,\nu,\sigma} P^{\rho\mu\nu\sigma}_{\tau_1\tau_2}(R_{\rho\mu\nu\sigma}+R_{\rho\nu\mu\sigma}) = aG_{\tau_1\tau_2} + bg_{\tau_1\tau_2}G,$$

where $a=-\alpha_1-\alpha_2+2\alpha_3+2\alpha_4-\alpha_5-\alpha_6-\beta_1-\beta_2+2\beta_3+2\beta_4-\beta_5-\beta_6$, and $b=2\alpha_9-\alpha_7-\alpha_8$. By Theorem 4 of §31 we have $N_{\tau_1\tau_2}=cg_{\tau_1\tau_2}$. Writing (μ,ν) for (τ_1,τ_2), (90) becomes

$$aG_{\mu\nu}+bg_{\mu\nu}G+cg_{\mu\nu}=0.$$

If $a=0$ this becomes

$$g_{\mu\nu}(bG+c)=0. \tag{95}$$

The solutions of this equation are the solutions of (93). Hence this gives us nothing new unless $a\neq 0$. Then we have

$$G_{\mu\nu}+g_{\mu\nu}(bG+c)=0. \tag{96}$$

Let us multiply this equation by $g^{\mu\nu}$ and sum as to μ,ν. Then

$$G+4(bG+c)=0.$$

Therefore

$$bG+c=-\tfrac{1}{4}G,$$

and (96) becomes

$$E_{\mu\nu}\equiv G_{\mu\nu}-\tfrac{1}{4}Gg_{\mu\nu}=0. \tag{97}$$

CASE 3. $m=2$.

By §31, $P^{\rho\mu\nu\sigma}_{\tau_1\tau_2\tau_3\tau_4}$ is a sum of 105 terms, each being a constant multiple of one of the three types

$$g^{\rho\mu}g^{\nu\sigma}g_{\tau_1\tau_2}g_{\tau_3\tau_4}\,; \quad g^{\rho\mu}g_{\tau_1\tau_2}I^{\nu}_{\tau_3}I^{\sigma}_{\tau_4}\,; \quad I^{\rho}_{\tau_1}I^{\mu}_{\tau_2}I^{\nu}_{\tau_3}I^{\sigma}_{\tau_4}.$$

The 105 terms correspond to all permutations of the sets of indices (ρ, μ, ν, σ) and $(\tau_1, \tau_2, \tau_3, \tau_4)$, which give distinct values to these three quantities. The terms of type

$$\sum_{\rho,\mu,\nu,\sigma} g^{\rho\mu}g^{\nu\sigma}g_{\tau_1\tau_2}g_{\tau_3\tau_4}(R_{\rho\mu\nu\sigma}+R_{\rho\nu\mu\sigma})$$

lead to

$$G(b_1 g_{\rho\mu}g_{\nu\sigma} + b_2 g_{\rho\nu}g_{\mu\sigma} + b_3 g_{\rho\sigma}g_{\mu\nu}).$$

The terms of type

$$\sum_{\rho,\mu,\nu,\sigma} g^{\rho\mu}g_{\tau_1\tau_2}I^{\nu}_{\tau_3}I^{\sigma}_{\tau_4}(R_{\rho\mu\nu\sigma}+R_{\rho\nu\mu\sigma})$$

lead to

$$a_1 G_{\rho\mu}g_{\nu\sigma} + a_2 G_{\rho\nu}g_{\mu\sigma} + a_3 G_{\rho\sigma}g_{\mu\nu} + a_4 G_{\mu\nu}g_{\rho\sigma} + a_5 G_{\mu\sigma}g_{\rho\nu} + a_6 G_{\nu\sigma}g_{\rho\mu}.$$

The terms of type

$$\sum_{\rho,\mu,\nu,\sigma} I^{\rho}_{\tau_1}I^{\mu}_{\tau_2}I^{\nu}_{\tau_3}I^{\sigma}_{\tau_4}(R_{\rho\mu\nu\sigma}+R_{\rho\nu\mu\sigma})$$

lead, in view of (50), to

$$a(R_{\rho\mu\nu\sigma} + R_{\rho\nu\mu\sigma}).$$

By Theorem 7 of §31

$$N_{\tau_1\tau_2\tau_3\tau_4} = c_1 g_{\tau_1\tau_2}g_{\tau_3\tau_4} + c_2 g_{\tau_1\tau_3}g_{\tau_2\tau_4} + c_3 g_{\tau_1\tau_4}g_{\tau_2\tau_3}.$$

Hence (90) becomes

$$\begin{aligned}(98)\quad & aR_{\rho\mu\nu\sigma} + aR_{\rho\nu\mu\sigma} + a_1 G_{\rho\mu}g_{\nu\sigma} + a_2 G_{\rho\nu}g_{\mu\sigma} + a_3 G_{\rho\sigma}g_{\mu\nu} + a_4 G_{\mu\nu}g_{\rho\sigma} \\ & + a_5 G_{\mu\sigma}g_{\rho\nu} + a_6 G_{\nu\sigma}g_{\rho\mu} + (b_1 G + c_1)g_{\rho\mu}g_{\nu\sigma} + (b_2 G + c_2)g_{\rho\nu}g_{\mu\sigma} \\ & + (b_3 G + c_3)g_{\rho\sigma}g_{\mu\nu} = 0.\end{aligned}$$

§37. *The Tensor* $W_{\rho\mu\nu\sigma}$.

The process of twice contracting the outer product of a covariant tensor $A_{m(>1)}$ and $g^{(2)}$ we shall call a *first reduction of* A_m *by* $g^{(2)}$. There are six first reductions of the left member of (98), corresponding to the indices of contraction (ρ, μ), (ρ, ν), (ρ, σ), (μ, ν), (μ, σ), (ν, σ); each reduction has the form $\alpha G_{(2)}+\beta g_{(2)}G+\gamma g_{(2)}$. We wish to know for what values of the constants in (98) every first reduction has its three coefficients α, β, γ all zero.

The reduction contracting by (ρ, μ) gives

$$\begin{aligned}G_{\nu\sigma}(-a + a_2 + a_3 + a_4 + a_5 + 4a_6) \\ + g_{\nu\sigma}G(a_1 + 4b_1 + b_2 + b_3) + g_{\nu\sigma}(4c_1 + c_2 + c_3).\end{aligned}$$

If we equate to zero the 18 coefficients of the six reductions we have

$$(99)\qquad \begin{cases} -a + a_2 + a_3 + a_4 + a_5 + 4a_6 = 0 \\ -a + a_1 + a_3 + a_4 + 4a_5 + a_6 = 0 \\ 2a + a_1 + a_2 + 4a_4 + a_5 + a_6 = 0 \\ 2a + a_1 + a_2 + 4a_3 + a_5 + a_6 = 0 \\ -a + a_1 + 4a_2 + a_3 + a_4 + a_6 = 0 \\ -a + 4a_1 + a_2 + a_3 + a_4 + a_5 = 0, \end{cases}$$

$$(100)\qquad \begin{cases} a_1 + 4b_1 + b_2 + b_3 = a_6 + 4b_1 + b_2 + b_3 = 0 \\ a_2 + b_1 + 4b_2 + b_3 = a_5 + b_1 + 4b_2 + b_3 = 0 \\ a_3 + b_1 + b_2 + 4b_3 = a_4 + b_1 + b_2 + 4b_3 = 0, \end{cases}$$

$$(101)\qquad \begin{cases} 4c_1 + c_2 + c_3 = 0 \\ c_1 + 4c_2 + c_3 = 0 \\ c_1 + c_2 + 4c_3 = 0. \end{cases}$$

Equations (99) give

$$\begin{cases} a_1 = a_2 = a_5 = a_6 = \dfrac{a}{2}, \\ a_3 = a_4 = -a; \end{cases}$$

now from (100)

$$b_1 = b_2 = -\frac{a}{6}, \quad b_3 = \frac{a}{3},$$

and by (101)

$$c_1 = c_2 = c_3 = 0.$$

Hence the left member of (98) is

$$a\big[R_{\rho\mu\nu\sigma} + R_{\rho\nu\mu\sigma} + \tfrac{1}{2}G_{\rho\mu}g_{\nu\sigma} + \tfrac{1}{2}G_{\rho\nu}g_{\mu\sigma} - G_{\rho\sigma}g_{\mu\nu} - G_{\mu\nu}g_{\rho\sigma} + \tfrac{1}{2}G_{\mu\sigma}g_{\rho\nu} + \tfrac{1}{2}G_{\nu\sigma}g_{\rho\mu} + \tfrac{1}{6}G(-g_{\rho\mu}g_{\nu\sigma} - g_{\rho\nu}g_{\mu\sigma} + 2g_{\rho\sigma}g_{\mu\nu})\big];$$

if now we define†

$$(102)\qquad W_{\rho\mu\nu\sigma} \equiv R_{\rho\mu\nu\sigma} + \tfrac{1}{2}G_{\rho\nu}g_{\mu\sigma} - \tfrac{1}{2}G_{\rho\sigma}g_{\mu\nu} - \tfrac{1}{2}G_{\mu\nu}g_{\rho\sigma} + \tfrac{1}{2}G_{\mu\sigma}g_{\rho\nu} + \tfrac{1}{6}G(g_{\rho\sigma}g_{\mu\nu} - g_{\rho\nu}g_{\mu\sigma}),$$

then the left member of (98) becomes

$$(103)\qquad a(W_{\rho\mu\nu\sigma} + W_{\rho\nu\mu\sigma}).$$

† This tensor was discovered by Weyl, who refers to it as the "conformal curvature tensor." H. Weyl: *Math. Zeitschrift*, (2), p. 404.

Properties of $W_{\rho\mu\nu\sigma}$.

It is readily verified that $W_{\rho\mu\nu\sigma}$ has the same symmetries (50) in ρ, μ, ν, σ as does $R_{\rho\mu\nu\sigma}$, and that in addition every first reduction of $W_{\rho\mu\nu\sigma}$ is identically zero, a property which $R_{\rho\mu\nu\sigma}$ does not satisfy (§27).

As will be seen in Chapter VII, $W_{\rho\mu\nu\sigma}$ is not identically zero.

There are at most 20 linearly independent components of this tensor; this follows from (50) just as in the case of $R_{\rho\mu\nu\sigma}$.

§38. *The Case* $m=m$.

By Theorem 13 of §31 we know that $P^{\rho\mu\nu\sigma}_{\tau_1,\cdots,\tau_{2m}}$ is a sum of terms each of which is a product of the factors $g_{(2)}$, $g^{(2)}$, and I; as in §36 we see that these terms are of three types:

$$g_{(2)} \overset{m-2}{\overbrace{\cdots}} g_{(2)}IIII, \quad g_{(2)} \overset{m-1}{\overbrace{\cdots}} g_{(2)}g^{(2)}II, \quad g_{(2)} \overset{m}{\overbrace{\cdots}} g_{(2)}g^{(2)}g^{(2)}.$$

Hence (90) may be written†

$$\text{(105)} \qquad \sum g_{(2)} \overset{m-2}{\overbrace{\cdots}} g_{(2)}(R_{(4)} + \breve{R}_{(4)}) + \sum g_{(2)} \overset{m-1}{\overbrace{\cdots}} g_{(2)}G_{(2)}$$
$$+ \sum g_{(2)} \overset{m}{\overbrace{\cdots}} g_{(2)}G + N_{(2m)} = 0,$$

or by (102) and (97)

$$\text{(106)} \qquad \sum g_{(2)} \overset{m-2}{\overbrace{\cdots}} g_{(2)}(W_{(4)} + \breve{W}_{(4)}) + \sum g_{(2)} \overset{m-1}{\overbrace{\cdots}} g_{(2)}E_{(2)}$$
$$+ GM_{(2m)} + N_{(2m)} = 0,$$

where each sum extends over all those permutations of the $2m$ subscripts giving linearly independent terms, and $\breve{R}_{\rho\mu\nu\sigma} \equiv R_{\rho\nu\mu\sigma}$, $\breve{W}_{\rho\mu\nu\sigma} \equiv W_{\rho\nu\mu\sigma}$.

LEMMA I. *If the constant obtained by* m *successive reductions of a fundamental non-differential tensor* $P_{(2m)}$ *by* $g^{(2)}$ *is zero then* $P_{(2m)}$ *is identically zero.* For if $P_{(2m)}$ is not identically zero then by Theorem I, Appendix II there is a first reduction $P_{(2m-2)}$ which is not identically zero; by the same theorem there is now a second reduction $P_{(2m-4)}$ not identically zero; hence by induction $P_{(0)}$ is not zero.

If now we reduce (106) m times by $g^{(2)}$ we obtain

$$GM_{(0)} + N_{(0)} = 0,$$

where $M_{(0)}$, $N_{(0)}$ are the mth reductions of $M_{(2m)}$, $N_{(2m)}$. If $M_{(0)}=0$ then $N_{(0)}=0$ and $M_{(2m)}$ and $N_{(2m)}$ are identically zero, by the lemma, so that (106) becomes

$$\text{(107)} \quad L_{2m} \equiv \sum g_{(2)} \overset{m-2}{\overbrace{\cdots}} g_{(2)}(W_{(4)} + \breve{W}_{(4)}) + \sum g_{(2)} \overset{m-1}{\overbrace{\cdots}} g_{(2)}E_{(2)}$$
$$\equiv A_{2m} + B_{2m} = 0.$$

† In (105) and in similar expressions we have omitted the constant coefficients in order to simplify the writing.

If $M_{(0)} \neq 0$ then $G=c$ and $GM_{(2m)}+N_{(2m)}=cM_{(2m)}+N_{(2m)}=P_{(2m)}$ is a fundamental non-differential tensor. But $P_{(0)}=cM_{(0)}+N_{(0)}=0$, so that $P_{(2m)}$ is identically zero, and (106) becomes (107) combined with (94). Hence the only laws are given by (107), and (107) combined with (94).

For the study of the laws (107) we shall find it convenient to introduce tensors

$$\overline{A}_{2m} \equiv \sum g_{(2)} \overset{m-2}{\overbrace{\cdots}} g_{(2)}(W_{(4)} + \breve{W}_{(4)}), \tag{108}$$

$$\overline{B}_{2m} \equiv \sum g_{(2)} \overset{m-1}{\overbrace{\cdots}} g_{(2)}E_{(2)}, \tag{109}$$

which have the property that all their first reductions are identically zero while the tensors themselves are not identically zero.

In §39 we shall show that there exist no $\overline{B}_{2m}$ except $\overline{B}_2$, and in §40 that there exist no $\overline{A}_{2m}$ except $\overline{A}_4$ and $\overline{A}_6$. In §42 we shall show that every law (107) is equivalent to

$$A_{2m} = 0 \tag{110}$$

with or without (97), so that every law of gravitation is equivalent to (110) with or without (97) and (94). In Appendix V it will be shown that (110) is equivalent to a set of laws

$$A_6 = 0, \tag{111}$$

and in §42 that (111) is equivalent to one or both of $\overline{A}_6=0$, $W_{(4)}=0$.

In §41 it will be shown that $\overline{A}_6=0$ is equivalent to $S_{(6)}=0$, where $S_{(6)}$ is given in (119). Consequently every possible law of gravitation will be known.

§39. *The Tensors* $\overline{B}_{2m}$.

For $m=1$ the only B_2 is $aE_{(2)}$; it follows from (97) that the first reduction of this is identically zero; hence it is a $\overline{B}_2$, and

$$\overline{B}_2 = aE_{(2)}. \tag{112}$$

For $m=2$ we have

$$\begin{aligned} B_4 = B_{\rho\mu\nu\sigma} = {} & a_1E_{\rho\mu}g_{\nu\sigma} + a_2E_{\rho\nu}g_{\mu\sigma} \\ & + a_3E_{\rho\sigma}g_{\mu\nu} + a_4E_{\mu\nu}g_{\rho\sigma} + a_5E_{\mu\sigma}g_{\rho\nu} + a_6E_{\nu\sigma}g_{\rho\mu}. \end{aligned} \tag{113}$$

If this is to be $\overline{B}_4$ then every one of its first reductions must be identically zero; since $E_{(2)}$ is not identically zero we are thus led to equations (99) with $a=0$; hence $a_i=0$ and there exists no $\overline{B}_4$.

THEOREM. *There exists no* $\overline{B}_{2m}$ *for* $m=2, 3, 4$.

We have proved this above for $m=2$; in Appendix III we prove it for $m=3, 4$.

PROPOSITION II. *There exists no $\overline{B}_{2m}$ for $m>4$.*

We have not been able to prove that this proposition is correct, but there are good reasons for thinking so (Appendix III).

§40. *The Tensors $\overline{A}_{2m}$.*

For $m=2$ the only A_4 is $a(W_{(4)}+\breve{W}_{(4)})$; it follows from the way in which (103) was derived that every first reduction of this is identically zero; hence it is an $\overline{A}_4$, and

$$\overline{A}_{\rho\mu\nu\sigma} = a(W_{\rho\mu\nu\sigma} + W_{\rho\nu\mu\sigma}). \tag{114}$$

We see readily, using (50), that

$$2\overline{A}_{\rho\mu\nu\sigma} + \overline{A}_{\rho\sigma\nu\mu} = 3aW_{\rho\mu\nu\sigma}\,; \tag{115}$$

hence $\overline{A}_4=0$ is equivalent to $W_{\rho\mu\nu\sigma}=0$.

Before treating the case $m=3$ we define the following set of fifteen functions and their conjugates:

$$\begin{aligned}
F_1(\alpha_1,\cdots,\alpha_{15}) \equiv F_1(\alpha_j) &\equiv \phi_1-2\alpha_4-2\alpha_5+\alpha_7+\alpha_8+\alpha_{10}+\alpha_{12}-2\alpha_{13}-2\alpha_{15}\\
\breve{F}_1(\alpha_j) &\equiv \phi_1+\alpha_4+\alpha_5-2\alpha_7-2\alpha_8-2\alpha_{10}-2\alpha_{12}+\alpha_{13}+\alpha_{15}\\
F_2(\alpha_j) &\equiv \phi_2-2\alpha_4-2\alpha_6+\alpha_7+\alpha_9-2\alpha_{10}-2\alpha_{11}+\alpha_{13}+\alpha_{14}\\
\breve{F}_2(\alpha_j) &\equiv \phi_2+\alpha_4+\alpha_6-2\alpha_7-2\alpha_9+\alpha_{10}+\alpha_{11}-2\alpha_{13}-2\alpha_{14}\\
F_3(\alpha_j) &\equiv \phi_3-2\alpha_5-2\alpha_6-2\alpha_8-2\alpha_9+\alpha_{11}+\alpha_{12}+\alpha_{14}+\alpha_{15}\\
&\cdots\\
F_4(\alpha_j) &\equiv \phi_4+\alpha_1+\alpha_2+\alpha_8+\alpha_9-2\alpha_{10}-2\alpha_{12}-2\alpha_{13}-2\alpha_{14}\\
&\cdots\\
F_5(\alpha_j) &\equiv \phi_5-2\alpha_1-2\alpha_3+\alpha_7+\alpha_8-2\alpha_{10}-2\alpha_{11}+\alpha_{14}+\alpha_{15}\\
&\cdots\\
F_6(\alpha_j) &\equiv \phi_6+\alpha_2+\alpha_3-2\alpha_7-2\alpha_9-2\alpha_{11}-2\alpha_{12}+\alpha_{13}+\alpha_{15}\\
&\cdots\\
F_7(\alpha_j) &\equiv \phi_7+\alpha_1+\alpha_2+\alpha_5+\alpha_6-2\alpha_{10}-2\alpha_{11}-2\alpha_{13}-2\alpha_{15}\\
&\cdots\\
F_8(\alpha_j) &\equiv \phi_8-2\alpha_1-2\alpha_3+\alpha_4+\alpha_5+\alpha_{11}+\alpha_{12}-2\alpha_{13}-2\alpha_{14}\\
&\cdots\\
F_9(\alpha_j) &\equiv \phi_9+\alpha_2+\alpha_3-2\alpha_4-2\alpha_6+\alpha_{10}+\alpha_{12}-2\alpha_{14}-2\alpha_{15}\\
&\cdots\\
F_{10}(\alpha_j) &\equiv \phi_{10}+\alpha_1+\alpha_2-2\alpha_4-2\alpha_5-2\alpha_7-2\alpha_9+\alpha_{14}+\alpha_{15}\\
&\cdots\\
F_{11}(\alpha_j) &\equiv \phi_{11}+\alpha_2+\alpha_3-2\alpha_5-2\alpha_6+\alpha_7+\alpha_8-2\alpha_{13}-2\alpha_{14}\\
&\cdots\\
F_{12}(\alpha_j) &\equiv \phi_{12}-2\alpha_1-2\alpha_3-2\alpha_4-2\alpha_6+\alpha_8+\alpha_9+\alpha_{13}+\alpha_{15}\\
&\cdots\\
F_{13}(\alpha_j) &\equiv \phi_{13}+\alpha_1+\alpha_2-2\alpha_4-2\alpha_6-2\alpha_7-2\alpha_8+\alpha_{11}+\alpha_{12}\\
&\cdots\\
F_{14}(\alpha_j) &\equiv \phi_{14}+\alpha_2+\alpha_3+\alpha_4+\alpha_5-2\alpha_8-2\alpha_9-2\alpha_{10}-2\alpha_{11}\\
&\cdots\\
F_{15}(\alpha_j) &\equiv \phi_{15}-2\alpha_1-2\alpha_3+\alpha_5+\alpha_6-2\alpha_7-2\alpha_9+\alpha_{10}+\alpha_{12}.
\end{aligned} \tag{116}$$

Here we have written explicitly only the first two of the conjugate functions; the others are formed in exactly the same manner. The ϕ_i are undetermined functions which are introduced to obtain more generality. We shall abbreviate this set of equations by writing it

$$F_i(\alpha_j) \equiv \phi_i + f_i(\alpha_j)$$
$$\breve{F}_i(\alpha_j) \equiv \phi_i + \breve{f}_i(\alpha_j).$$

We then have

LEMMA I. *The set of equations*

$$F_i(\alpha_j) - \breve{F}_i(\alpha_j) = f_i(\alpha_j) - \breve{f}_i(\alpha_j) = 0$$

is of rank 9, and its most general solution is

$$\begin{aligned}
\alpha_2 &= -\alpha_3 + \alpha_9 + \alpha_{11} \\
\alpha_5 &= \alpha_1 - \alpha_4 + \alpha_{14} \\
\alpha_6 &= \alpha_9 + \alpha_{11} - \alpha_{14} \\
\alpha_7 &= \alpha_1 - \alpha_3 - \alpha_4 + \alpha_9 + \alpha_{11} \\
\alpha_8 &= \alpha_3 + \alpha_4 - \alpha_9 \\
\alpha_{10} &= \alpha_1 - \alpha_3 - \alpha_4 + \alpha_9 + \alpha_{14} \\
\alpha_{12} &= \alpha_3 + \alpha_4 - \alpha_{14} \\
\alpha_{13} &= \alpha_4 + \alpha_{11} - \alpha_{14} \\
\alpha_{15} &= \alpha_1 - \alpha_4 + \alpha_9,
\end{aligned}$$

where α_1, α_3, α_4, α_9, α_{11}, *and* α_{14} *are arbitrary.*

This fact is easily verified and will not be proved here.

LEMMA II. *If* $\phi_i = 4\alpha_i$ *then the set of 30 equations*

$$F_i(\alpha_j) = 0$$
$$\breve{F}_i(\alpha_j) = 0$$

admits the unique solution $\alpha_i = \alpha$, *where* α *is arbitrary.*

This result is obtained by putting $\phi_i = 4\alpha_i$ in the equations $F_i(\alpha_j) = 0$, $\breve{F}_i(\alpha_j) = 0$, and substituting in them the partial solution given in Lemma I.

We are now ready to consider the case $m = 3$. Let us represent the term $g_{\mu_5\mu_6}\alpha_{1234}(W_{\mu_1\mu_2\mu_3\mu_4} + W_{\mu_1\mu_3\mu_2\mu_4})$ of (107) by the symbol $56-1234$; by (107), A_6 is the sum of $C_4^6 = 15$ such terms with distinct indices. If this is to be an $\overline{A}_6$ then every one of its fifteen first reductions must vanish identically; this condition furnishes us equations in the α; in order that these equations have certain symmetry properties which make them readily solvable it is necessary to choose the order of the indices in a particular manner as

follows: We take A_6 equal to the sum of the following fifteen terms, each column of three terms being obtained from the preceding column by a substitution operation on pairs:

$$
(117)\qquad \begin{array}{lllll}
56-1342 & 36-1254 & 45-1263 & 35-1246 & 46-1235 \\
34-5126 & 25-3146 & 26-4135 & 24-3165 & 23-4156 \\
12-3564 & 14-2365 & 13-2456 & 16-2354 & 15-2463.
\end{array}
$$

If we reduce by $g^{\lambda\tau}$ the term $\lambda\tau - \cdots$ is called the *base*. Now if, for example, we reduce by g^{56} we obtain terms which may be reduced, by (50), into *two sets*, those in W_{1342} and those in its conjugate W_{1432}. On reduction by $g^{\lambda\tau}$ the base gives four times its α in each of the two sets of terms; if both of the reducing indices are on the right of the term in (117) we get zero (§37); if one of the reducing indices is on the left and the other on the right we obtain the W and $\bar{W}$ of the base. For example, if we reduce by g^{56} the base is $56-1342$; this term gives $4\alpha_{1342}(W_{1342}+W_{1432})$; the term $34-5126$ gives zero; the term $36-1254$ gives $\alpha_{1254}(W_{1234}+W_{1324})$ which, by (50), is $\alpha_{1254}(-2W_{1342}+W_{1432})$. We may note that for each pair of reducing indices exactly nine terms of (117) have non-zero reductions.

In the reduction equations obtained by reducing (117) three types of terms occur:

Type 1: The numerical coefficients of $\alpha_{\rho\mu\nu\sigma}$ in the equation and its conjugate are respectively $+1$ and -2.

Type 2: The numerical coefficients of $\alpha_{\rho\mu\nu\sigma}$ in the equation and its conjugate are respectively -2 and $+1$.

Type 3: The numerical coefficient of $\alpha_{\rho\mu\nu\sigma}$ is the same in both the equation and its conjugate.

The fundamental property of the table (117) is that terms of Type 3 enter only from the base. Also in each reduction equation there are exactly four terms of each of Types 1 and 2.

Let us denote the α's by their order in (117):

$$\alpha_{1342}\equiv\alpha_1,\quad \alpha_{5126}\equiv\alpha_2,\quad\cdots,\quad \alpha_{2463}\equiv\alpha_{15}.$$

Now if we reduce (117) in the $C_2^6=15$ possible ways and equate them to zero we have the thirty equations (116) with $\phi_\iota=4\alpha_\iota$; by Lemma II these equations have only the solution $\alpha_\iota=\alpha$, an arbitrary constant; hence there exists only one $\bar{A}_6$:

$$
\begin{aligned}
\frac{1}{\alpha}\bar{A}_{\sigma\tau\rho\mu\nu\lambda} &= g_{\sigma\tau}(W_{\rho\mu\nu\lambda}+W_{\rho\nu\mu\lambda})+g_{\sigma\rho}(W_{\lambda\nu\tau\mu}+W_{\lambda\tau\nu\mu})+g_{\sigma\mu}(W_{\lambda\rho\tau\nu}+W_{\lambda\tau\rho\nu}) \\
&\quad+g_{\sigma\nu}(W_{\rho\lambda\tau\mu}+W_{\rho\tau\lambda\mu})+g_{\sigma\lambda}(W_{\mu\rho\nu\tau}+W_{\mu\nu\rho\tau})+g_{\tau\rho}(W_{\mu\lambda\nu\sigma}+W_{\mu\nu\lambda\sigma}) \\
(118)\quad &\quad+g_{\tau\mu}(W_{\rho\lambda\sigma\nu}+W_{\rho\sigma\lambda\nu})+g_{\tau\nu}(W_{\rho\lambda\mu\sigma}+W_{\rho\mu\lambda\sigma})+g_{\tau\lambda}(W_{\nu\rho\mu\sigma}+W_{\nu\mu\rho\sigma}) \\
&\quad+g_{\rho\mu}(W_{\lambda\nu\sigma\tau}+W_{\lambda\sigma\nu\tau})+g_{\rho\nu}(W_{\lambda\mu\tau\sigma}+W_{\lambda\tau\mu\sigma})+g_{\rho\lambda}(W_{\mu\sigma\tau\nu}+W_{\mu\tau\sigma\nu}) \\
&\quad+g_{\mu\nu}(W_{\sigma\rho\lambda\tau}+W_{\sigma\lambda\rho\tau})+g_{\mu\lambda}(W_{\nu\rho\sigma\tau}+W_{\nu\sigma\rho\tau})+g_{\nu\lambda}(W_{\rho\mu\sigma\tau}+W_{\rho\sigma\mu\sigma}).
\end{aligned}
$$

THEOREM. *There exists no* $\overline{A}_8$.

This is proved in Appendix IV.

PROPOSITION III. *There exists no* $\overline{A}_{2m}$ *for* $m>4$.

We have not proved the correctness of this, but have good reasons for assuming it (Appendix IV).

§41. *The Tensor* $S_{\sigma\tau\rho\mu\nu\lambda}$.

Let us define a new tensor

$$
\begin{aligned}
S_{\sigma\tau\rho\mu\nu\lambda} \equiv\ & g_{\sigma\tau}W_{\rho\mu\nu\lambda} + g_{\sigma\rho}W_{\mu\lambda\nu\tau} + g_{\sigma\mu}W_{\lambda\rho\nu\tau} + g_{\sigma\nu}W_{\mu\rho\tau\lambda} + g_{\tau\rho}W_{\sigma\lambda\mu\nu} \\
& + g_{\tau\mu}W_{\lambda\sigma\rho\nu} + g_{\tau\lambda}W_{\rho\mu\sigma\nu} + g_{\rho\nu}W_{\mu\tau\lambda\sigma} + g_{\rho\lambda}W_{\mu\sigma\tau\nu} + g_{\mu\nu}W_{\sigma\lambda\rho\tau} \\
& + g_{\mu\lambda}W_{\rho\sigma\nu\tau} + g_{\nu\lambda}W_{\rho\mu\tau\sigma}.
\end{aligned}
\tag{119}
$$

Then it is readily verified, using (50), that

$$\frac{1}{3\alpha}(\overline{A}_{\sigma\tau\rho\mu\nu\lambda} - \overline{A}_{\sigma\tau\rho\nu\mu\lambda}) = S_{\mu\tau\rho\lambda\sigma\nu}, \tag{120}$$

$$\frac{1}{\alpha}(\overline{A}_{\sigma\tau\rho\mu\nu\lambda} + \overline{A}_{\sigma\tau\rho\nu\mu\lambda}) = S_{\sigma\tau\rho\mu\nu\lambda} + S_{\sigma\tau\rho\nu\mu\lambda} + S_{\sigma\tau\lambda\mu\nu\rho} + S_{\sigma\tau\lambda\nu\mu\rho}. \tag{121}$$

Hence

$$\overline{A}_{\sigma\tau\rho\mu\nu\lambda} = \frac{\alpha}{2}(3S_{\mu\tau\rho\lambda\sigma\nu} + S_{\sigma\tau\rho\mu\nu\lambda} + S_{\sigma\tau\rho\nu\mu\lambda} + S_{\sigma\tau\lambda\mu\nu\rho} + S_{\sigma\tau\lambda\nu\mu\rho}). \tag{122}$$

Properties of $S_{\sigma\tau\rho\mu\nu\lambda}$.

By equations (120) and (122) it follows at once that $S_{(6)}=0$ is equivalent to $\overline{A}_6=0$. By (120) it follows that every first reduction of $S_{(6)}$ is identically zero, since $\overline{A}_6$ satisfies this property. Using (50) we can directly verify that $S_{(6)}$ has the following symmetries:

$$S_{\alpha\alpha\alpha\mu\nu\lambda} = S_{\alpha\alpha\rho\alpha\nu\lambda} = S_{\alpha\alpha\rho\mu\alpha\lambda} = \cdots = S_{\sigma\tau\rho\alpha\alpha\alpha} = 0, \tag{123}$$

i.e., if any 3 of the subscripts are equal, then that component is identically zero.

$$\begin{cases} S_{\sigma\tau\rho\mu\nu\lambda} = -S_{\lambda\nu\mu\rho\tau\sigma}, \\ S_{\sigma\tau\rho\mu\nu\lambda} = -S_{\sigma\tau\mu\rho\nu\lambda}, \end{cases} \tag{124}$$

$$\begin{cases} S_{\sigma\tau\rho\mu\nu\sigma} + S_{\sigma\tau\rho\nu\sigma\mu} + S_{\sigma\tau\rho\sigma\mu\nu} = 0, \\ S_{\lambda\nu\rho\mu\tau\sigma} + S_{\nu\mu\rho\lambda\tau\sigma} + S_{\mu\lambda\rho\nu\tau\sigma} = 0; \end{cases} \tag{125}$$

this last follows at once from the 3 preceeding ones. The following 15 quantities have the same symmetries in ρ, μ, ν, σ as in (50):

$$(126)\qquad \begin{cases} S_{\alpha\alpha\rho\mu\nu\sigma},\ S_{\alpha\rho\alpha\nu\mu\sigma},\ S_{\alpha\rho\nu\alpha\mu\sigma},\ S_{\alpha\rho\nu\sigma\alpha\mu},\ S_{\alpha\rho\mu\nu\sigma\alpha}, \\ S_{\rho\alpha\alpha\nu\sigma\mu},\ S_{\rho\alpha\nu\alpha\sigma\mu},\ S_{\rho\alpha\sigma\mu\alpha\nu},\ S_{\rho\alpha\sigma\nu\mu\alpha}, \\ S_{\rho\mu\alpha\alpha\nu\sigma},\ S_{\rho\nu\alpha\sigma\alpha\mu},\ S_{\rho\nu\alpha\mu\sigma\alpha}, \\ S_{\rho\nu\sigma\alpha\alpha\mu},\ S_{\rho\nu\mu\alpha\sigma\alpha}, \\ S_{\rho\mu\nu\sigma\alpha\alpha}\ ; \end{cases}$$

the last five of these follow directly from the first seven by (124), the 15th from the 1st, the 14th from the 2nd, the 13th from the 6th, the 12th from the 3rd, the 11th from the 7th.

In addition,

$$(127)\qquad S_{\alpha\rho\mu\nu\sigma\alpha} = S_{\rho\alpha\mu\nu\alpha\sigma} = S_{\rho\mu\alpha\alpha\nu\sigma} = 0,$$

$$(128)\qquad \begin{cases} S_{\alpha\beta\alpha\beta\gamma\delta} = S_{\alpha\beta\beta\gamma\alpha\delta} = -S_{\alpha\alpha\beta\gamma\beta\delta} = -S_{\beta\beta\alpha\gamma\alpha\delta}, \\ S_{\alpha\alpha\beta\gamma\beta\gamma} = S_{\beta\beta\alpha\gamma\alpha\gamma} = S_{\gamma\gamma\alpha\beta\alpha\beta}. \end{cases}$$

From these symmetries (123−128) it follows that of the $4^6=4{,}096$ components of $S_{(6)}$ there are at most 10 linearly independent ones, given by the subscripts

112323	112324	223134
112424	113234	224143
113434	114243	334142
223434.		

§42. *The Laws* $L_{2m}=0$.

Assuming the correctness of the Propositions in §§39 and 40 we know that there exist no $\overline{A}_{2m}$, $\overline{B}_{2m}$ except

$$(129)\qquad \begin{cases} \overline{B}_2 \\ \overline{A}_4 \\ \overline{A}_6. \end{cases}$$

We shall now consider the general law with $m \geqq 3$,

$$(130)\qquad L_{2m} \equiv A_{2m} + B_{2m} = 0.$$

By §39 if B_{2m} is not identically zero one of its first reductions is not identically zero; by induction, at least one of its $(m-1)$st reductions is not identically zero and hence is a non-zero constant multiple of $E_{(2)}$; but every $(m-1)$st reduction of A_{2m} is identically zero; hence there is an $(m-1)$st reduction of L_{2m} which is a non-zero constant multiple of $E_{(2)}$; hence (130) implies $E_{(2)}=0$ and hence $B_{2m}=0$ and $A_{2m}=0$. But $E_{(2)}=A_{2m}=0$ implies $L_{2m}=0$; hence $L_{2m}=0$ is equivalent to $A_{2m}=0$ and $E_{(2)}=0$ combined. If

B_{2m} is identically zero then $L_{2m}=0$ is equivalent to $A_{2m}=0$. Hence every law (130) is equivalent to

$$A_{2m} = 0 \tag{131}$$

with or without (97).

Now consider (131) where A_{2m} is not identically zero. By Appendix V $A_{2m}=0$ is equivalent to the set of equations obtained by equating to zero all the $(m-3)$rd reductions of A_{2m}. Hence every possible law (107) is equivalent to a set of laws $A_6=0$, with or without (97) and (94).

Now consider

$$A_6 = 0, \tag{132}$$

where A_6 is not identically zero. If $A_6=\overline{A}_6$ then (132) is $\overline{A}_6=0$, which is equivalent to $S_{(6)}=0$. If $A_6\neq\overline{A}_6$ then one of the first reductions of A_6 is not identically zero and must be

$$aW_{(4)} + b\breve{W}_{(4)}, \tag{133}$$

where a, $b\neq 0$, 0. Hence (132) implies

$$A_{\rho\mu\nu\sigma} = aW_{\rho\mu\nu\sigma} + bW_{\rho\nu\mu\sigma} = 0.$$

But

$$2A_{\rho\mu\nu\sigma} + 2A_{\rho\sigma\nu\mu} + A_{\rho\nu\mu\sigma} + A_{\rho\sigma\mu\nu} = 3aW_{\rho\mu\nu\sigma}\,;$$

hence

$$aW_{\rho\mu\nu\sigma} = bW_{\rho\nu\mu\sigma} = 0\,;$$

but a, $b\neq 0$, 0; hence (132) is equivalent to

$$W_{\rho\mu\nu\sigma} = 0. \tag{134}$$

§43. *The Laws of Gravitation.*

We have shown in §42 that every possible law of gravitation is equivalent to one or more of (94), (97), $W_{(4)}=0$, $S_{(6)}=0$. Hence *the only possible laws of gravitation are*

$$\begin{aligned} G &= c, \\ E_{\mu\nu} &= 0, \\ W_{\rho\mu\nu\sigma} &= 0, \\ S_{\lambda\tau\rho\mu\nu\sigma} &= 0, \end{aligned}$$

and the combinations of 2 *or more of these.*

§44. *The Divergence of Certain Tensors.*

For a tensor $A_{\mu_1 \ldots \mu_m}$ (h_μ) let

$$(135) \qquad A^{\nu}_{\mu_1 \cdots \mu_{m-1}} \equiv \sum_{\mu_m} g^{\mu_m \nu} A_{\mu_1 \cdots \mu_m} ;$$

then by §11

$$A_{\mu_1 \cdots \mu_m} = 0 \text{ is equivalent to } A^{\nu}_{\mu_1 \cdots \mu_{m-1}} = 0.$$

By §21 we have

$$(136) \qquad \nabla_\nu A^{\nu}_{\mu_1 \cdots \mu_{m-1}} = \sum_{\nu} \frac{\mathfrak{D} A^{\nu}_{\mu_1 \cdots \mu_{m-1}}}{\mathfrak{D} x_\nu}.$$

It follows from §17 that

$$\nabla_\nu A^{\nu}_{\ldots} \pm \nabla_\nu B^{\nu}_{\ldots} = \nabla_\nu (A^{\nu}_{\ldots} \pm B^{\nu}_{\ldots}) .$$

From (34) it follows that *at the origin of canonical coördinates* of §28 we have

$$\frac{\mathfrak{D} A^{\nu}_{\mu_1 \cdots \mu_{m-1}}}{\mathfrak{D} x_\nu} = \frac{\partial A^{\nu}_{\mu_1 \cdots \mu_{m-1}}}{\partial x_\nu},$$

and hence

$$(137) \qquad \nabla_\nu A^{\nu}_{\mu_1 \cdots \mu_{m-1}} = \sum_{\nu} \frac{\partial A^{\nu}_{\mu_1 \cdots \mu_{m-1}}}{\partial x_\nu} .$$

We shall now consider the value of the divergence of several of the foregoing tensors *at the origin of canonical coördinates.*

By (135)

$$G^{\nu}_{\mu} = \sum_{\sigma} g^{\nu\sigma} G_{\mu\sigma} = \sum_{\sigma,\rho,\tau} g^{\nu\sigma} g^{\rho\tau} R_{\mu\rho\tau\sigma} .$$

Hence, by the first of (58),

$$\nabla_\nu G^{\nu}_{\mu} = \sum_{\nu,\sigma,\rho,\tau} g^{\nu\sigma} g^{\rho\tau} \frac{\partial R_{\mu\rho\tau\sigma}}{\partial x_\nu} .$$

Now by (49)

$$(138) \quad \nabla_\nu G^{\nu}_{\mu} = \frac{1}{2} \sum_{\nu,\rho,\sigma,\tau} g^{\nu\sigma} g^{\rho\tau} \frac{\partial}{\partial x_\nu} \left(\frac{\partial^2 g_{\mu\sigma}}{\partial x_\rho \partial x_\tau} + \frac{\partial^2 g_{\rho\tau}}{\partial x_\mu \partial x_\sigma} - \frac{\partial^2 g_{\mu\tau}}{\partial x_\rho \partial x_\sigma} - \frac{\partial^2 g_{\rho\sigma}}{\partial x_\mu \partial x_\tau} \right)$$

$$= \frac{1}{2} \sum_{\nu,\rho,\sigma,\tau} g^{\nu\sigma} g^{\rho\tau} \left(\frac{\partial^3 g_{\mu\sigma}}{\partial x_\nu \partial x_\rho \partial x_\tau} - \frac{\partial^3 g_{\mu\tau}}{\partial x_\nu \partial x_\rho \partial x_\sigma} \right)$$

$$+ \frac{1}{2} \sum_{\nu,\rho,\sigma,\tau} g^{\nu\sigma} g^{\rho\tau} \left(\frac{\partial^3 g_{\rho\tau}}{\partial x_\nu \partial x_\mu \partial x_\sigma} - \frac{\partial^3 g_{\rho\sigma}}{\partial x_\nu \partial x_\mu \partial x_\tau} \right).$$

By interchanging ρ with ν, and τ with σ we find

$$\frac{1}{2}\sum_{\nu,\rho,\sigma,\tau} g^{\nu\sigma}g^{\rho\tau}\left(\frac{\partial^3 g_{\mu\sigma}}{\partial x_\nu \partial x_\rho \partial x_\tau} - \frac{\partial^3 g_{\mu\tau}}{\partial x_\nu \partial x_\rho \partial x_\sigma}\right)$$
$$= \frac{1}{2}\sum_{\nu,\rho,\sigma,\tau} g^{\rho\tau}g^{\nu\sigma}\left(\frac{\partial^3 g_{\mu\tau}}{\partial x_\rho \partial x_\nu \partial x_\sigma} - \frac{\partial^3 g_{\mu\sigma}}{\partial x_\rho \partial x_\nu \partial x_\tau}\right).$$

Since this quantity is thus its own negative it must be zero. Thus (138) becomes

$$\nabla_\nu G_\mu^\nu = \frac{1}{2}\sum_{\nu,\rho,\sigma,\tau} g^{\nu\sigma}g^{\rho\tau}\left(\frac{\partial^3 g_{\rho\tau}}{\partial x_\nu \partial x_\mu \partial x_\sigma} - \frac{\partial^3 g_{\rho\sigma}}{\partial x_\nu \partial x_\mu \partial x_\tau}\right). \tag{139}$$

Similarly

$$\nabla_\nu G g_\mu^\nu = \sum_\nu g_\mu^\nu \frac{\partial G}{\partial x_\nu} = \frac{\partial G}{\partial x_\mu} = \sum_{\nu,\rho,\sigma,\tau} g^{\nu\sigma}g^{\rho\tau}\frac{\partial R_{\nu\rho\tau\sigma}}{\partial x_\mu}.$$

If we treat this as we did (138) we find

$$\nabla_\nu G g_\mu^\nu = \frac{\partial G}{\partial x_\mu} = \sum_{\nu,\rho,\sigma,\tau} g^{\nu\sigma}g^{\rho\tau}\left(\frac{\partial^3 g_{\rho\tau}}{\partial x_\mu \partial x_\nu \partial x_\sigma} - \frac{\partial^3 g_{\rho\sigma}}{\partial x_\mu \partial x_\nu \partial x_\tau}\right).$$

Comparing this with (139), we see that

$$\nabla_\nu G_\mu^\nu = \frac{1}{2}\frac{\partial G}{\partial x_\mu}, \tag{140}$$

$$\nabla_\nu G g_\mu^\nu = \frac{\partial G}{\partial x_\mu}. \tag{141}$$

We also have

$$\frac{\partial G_{\mu\nu}}{\partial x_\rho} = \sum_{\sigma,\tau} g^{\sigma\tau}\frac{\partial R_{\sigma\mu\nu\tau}}{\partial x_\rho} = \frac{1}{2}\sum_{\sigma,\tau} g^{\sigma\tau}\left(\frac{\partial^3 g_{\sigma\tau}}{\partial x_\mu \partial x_\nu \partial x_\rho}\right.$$
$$\left. + \frac{\partial^3 g_{\mu\nu}}{\partial x_\sigma \partial x_\tau \partial x_\rho} - \frac{\partial^3 g_{\sigma\nu}}{\partial x_\mu \partial x_\tau \partial x_\rho} - \frac{\partial^3 g_{\mu\tau}}{\partial x_\sigma \partial x_\nu \partial x_\rho}\right),$$

$$\frac{\partial G_{\rho\nu}}{\partial x_\mu} = \frac{1}{2}\sum_{\sigma,\tau} g^{\sigma\tau}\left(\frac{\partial^3 g_{\sigma\tau}}{\partial x_\rho \partial x_\nu \partial x_\mu} + \frac{\partial^3 g_{\rho\nu}}{\partial x_\sigma \partial x_\tau \partial x_\mu} - \frac{\partial^3 g_{\sigma\nu}}{\partial x_\rho \partial x_\tau \partial x_\mu} - \frac{\partial^3 g_{\rho\tau}}{\partial x_\sigma \partial x_\nu \partial x_\mu}\right).$$

Hence

$$\frac{\partial G_{\mu\nu}}{\partial x_\rho} - \frac{\partial G_{\rho\nu}}{\partial x_\mu} = \frac{1}{2}\sum_{\sigma,\tau} g^{\sigma\tau}\left(\frac{\partial^3 g_{\mu\nu}}{\partial x_\sigma \partial x_\tau \partial x_\rho} + \frac{\partial^3 g_{\rho\tau}}{\partial x_\sigma \partial x_\mu \partial x_\nu} - \frac{\partial^3 g_{\mu\tau}}{\partial x_\sigma \partial x_\nu \partial x_\rho} - \frac{\partial^3 g_{\rho\nu}}{\partial x_\sigma \partial x_\tau \partial x_\mu}\right)$$
$$= \sum_{\sigma,\tau} g^{\sigma\tau}\frac{\partial R_{\rho\mu\nu\tau}}{\partial x_\sigma}.$$

$$\nabla_\sigma R_{\rho\mu\nu}^\sigma = \sum_\sigma \frac{\partial R_{\rho\mu\nu}^\sigma}{\partial x_\sigma} = \sum_{\sigma\,\tau} g^{\sigma\tau}\frac{\partial R_{\rho\mu\nu\tau}}{\partial x_\sigma}.$$

Hence

(142) $$\nabla_\sigma R^{\sigma}_{\rho\mu\nu} = \frac{\partial G_{\mu\nu}}{\partial x_\rho} - \frac{\partial G_{\rho\nu}}{\partial x_\mu}.$$

Also

(143) $$\nabla_\sigma G_{\rho\mu} g_\nu{}^\sigma = \sum_\sigma g_\nu{}^\sigma \frac{\partial G_{\rho\mu}}{\partial x_\sigma} = \frac{\partial G_{\rho\mu}}{\partial x_\nu},$$

(144) $$\nabla_\sigma G_\rho{}^\sigma g_{\mu\nu} = g_{\mu\nu} \nabla_\sigma G_\rho{}^\sigma = \frac{1}{2} g_{\mu\nu} \frac{\partial G}{\partial x_\rho},$$

(145) $$\nabla_\sigma G g_{\rho\mu} g_\nu{}^\sigma = g_{\rho\mu} \nabla_\sigma G g_\nu{}^\sigma = g_{\rho\mu} \frac{\partial G}{\partial x_\nu}.$$

From the preceding relations we find, since

$$\frac{\mathfrak{D} G_{\mu\nu}}{\mathfrak{D} x_\rho} = \frac{\partial G_{\mu\nu}}{\partial x_\rho},$$

(146) $$\nabla_\nu E_\mu^\nu = \frac{1}{4} \frac{\partial G}{\partial x_\mu},$$

(147) $$\nabla_\sigma W^{\sigma}_{\rho\mu\nu} = \frac{1}{2}\left(\frac{\mathfrak{D} G_{\mu\nu}}{\mathfrak{D} x_\rho} - \frac{\mathfrak{D} G_{\rho\nu}}{\mathfrak{D} x_\mu}\right) - \frac{1}{12}\left(g_{\mu\nu} \frac{\partial G}{\partial x_\rho} - g_{\rho\nu} \frac{\partial G}{\partial x_\mu}\right),$$

(148) $$\nabla_\sigma R^{\sigma}_{\rho\mu\nu} = \frac{\mathfrak{D} G_{\mu\nu}}{\mathfrak{D} x_\rho} - \frac{\mathfrak{D} G_{\rho\nu}}{\mathfrak{D} x_\mu}.$$

It is to be noted that we have shown these relations only at the origin of canonical coördinates.

By II of §18 we know that $\partial G/\partial x_\mu$ is a tensor, and by (34) we know that $\mathfrak{D} G_{\mu\nu}/\mathfrak{D} x_\rho$ is a tensor. Hence the relations (146–148) are between tensors. Hence by (60) *the relations* (146–148) *are true at every point of every coördinate system.*

Another relation, that of *Bianchi*, is easily proved:

(149) $$\frac{\mathfrak{D} R_{\mu\sigma\nu\tau}}{\mathfrak{D} x_\rho} + \frac{\mathfrak{D} R_{\sigma\rho\nu\tau}}{\mathfrak{D} x_\mu} + \frac{\mathfrak{D} R_{\rho\mu\nu\tau}}{\mathfrak{D} x_\sigma} = 0;$$

for at the origin of canonical coördinates this becomes

$$\frac{\partial R_{\mu\sigma\nu\tau}}{\partial x_\rho} + \frac{\partial R_{\sigma\rho\nu\tau}}{\partial x_\mu} + \frac{\partial R_{\rho\mu\nu\tau}}{\partial x_\sigma} = 0,$$

which is verified directly by using (49); hence the relation (149) is true at every point. Similarly it is found that

(150) $$\frac{\mathfrak{D} R_{\rho\mu\sigma\nu}}{\mathfrak{D} x_\tau} + \frac{\mathfrak{D} R_{\mu\sigma\tau\nu}}{\mathfrak{D} x_\rho} + \frac{\mathfrak{D} R_{\rho\sigma\nu\tau}}{\mathfrak{D} x_\mu} + \frac{\mathfrak{D} R_{\rho\mu\tau\sigma}}{\mathfrak{D} x_\nu} = 0.$$

It is now found readily, using (119), (135), (147), and (102) that

$$\nabla_\lambda S^{\lambda}_{\sigma\tau\rho\mu\nu} = 0 \tag{151}$$

is an identity.

§45. *Applications.*

Consider a tensor equation

$$A^{\nu}_{\mu_1\cdots\mu_{m-1}} = 0. \tag{152}$$

On every solution of (152) we have also

$$\frac{\partial A^{\nu}_{\mu_1\cdots\mu_{m-1}}}{\partial x_\nu} = 0,$$

so that by (137) at the origin of canonical coördinates we have

$$\nabla_\nu A^{\nu}_{\mu_1\cdots\mu_{m-1}} = 0\,; \tag{153}$$

since the left member of this equation is a tensor the equation holds at every point. Hence *every solution of* (152) *is also a solution of* (153).

Now if (97) is true then by (146) we have

$$\frac{\partial G}{\partial x_\mu} = 0,$$

so that $G = \text{const.}$

Hence by §43:

With the 5 *Postulates of* §34 *and the* 3 *Propositions of* §§34, 39, 40 *the only possible laws of gravitation are*

$$G = c\,; \tag{94}$$

$$G_{\mu\nu} - kg_{\mu\nu} = 0, \quad (k = \tfrac{1}{4}G = \text{const.}) \tag{154}$$

$$\begin{aligned} W_{\rho\mu\nu\sigma} \equiv R_{\rho\mu\nu\sigma} &+ \tfrac{1}{2}G_{\rho\nu}g_{\mu\sigma} - \tfrac{1}{2}G_{\rho\sigma}g_{\mu\nu} - \tfrac{1}{2}G_{\mu\nu}g_{\rho\sigma} + \tfrac{1}{2}G_{\mu\sigma}g_{\rho\nu} \\ &+ \tfrac{1}{6}G(g_{\rho\sigma}g_{\mu\nu} - g_{\rho\nu}g_{\mu\sigma}) = 0, \end{aligned} \tag{134}$$

$$\begin{aligned} S_{\sigma\tau\rho\mu\nu\lambda} \equiv\; & g_{\sigma\tau}W_{\rho\mu\nu\lambda} + g_{\sigma\rho}W_{\mu\lambda\nu\tau} + g_{\sigma\mu}W_{\lambda\rho\nu\tau} + g_{\sigma\nu}W_{\mu\rho\tau\lambda} + g_{\tau\rho}W_{\sigma\lambda\mu\nu} \\ &+ g_{\tau\mu}W_{\lambda\sigma\rho\nu} + g_{\tau\lambda}W_{\rho\mu\sigma\nu} + g_{\rho\nu}W_{\mu\tau\lambda\sigma} + g_{\rho\lambda}W_{\mu\sigma\tau\nu} + g_{\mu\nu}W_{\sigma\lambda\rho\tau} \\ &+ g_{\mu\lambda}W_{\rho\sigma\nu\tau} + g_{\nu\lambda}W_{\rho\mu\sigma\tau} = 0, \end{aligned} \tag{155}$$

$$W_{\rho\mu\nu\sigma} = 0 \text{ with } G = c, \tag{156}$$

$$S_{\sigma\tau\rho\mu\nu\lambda} = 0 \text{ with } G = c, \tag{157}$$

$$R_{\rho\mu\nu\sigma} + \frac{k}{3}(g_{\rho\nu}g_{\mu\sigma} - g_{\rho\sigma}g_{\mu\nu}) = 0, \tag{158}$$

$$
\begin{aligned}
(159)\quad & g_{\sigma\tau}R_{\rho\mu\nu\lambda} + g_{\sigma\rho}R_{\mu\lambda\nu\tau} + g_{\sigma\mu}R_{\lambda\rho\nu\tau} + g_{\sigma\nu}R_{\mu\rho\tau\lambda} + g_{\tau\rho}R_{\sigma\lambda\mu\nu} + g_{\tau\mu}R_{\lambda\sigma\rho\nu} \\
& + g_{\tau\lambda}R_{\rho\mu\sigma\nu} + g_{\rho\nu}R_{\mu\tau\lambda\sigma} + g_{\rho\lambda}R_{\mu\sigma\tau\nu} + g_{\mu\nu}R_{\sigma\lambda\rho\tau} + g_{\mu\lambda}R_{\rho\sigma\nu\tau} + g_{\nu\lambda}R_{\rho\mu\sigma\tau} \\
& + k\big[g_{\sigma\tau}(g_{\rho\nu}g_{\mu\lambda} - g_{\rho\lambda}g_{\mu\nu}) + g_{\sigma\rho}(g_{\mu\nu}g_{\lambda\tau} - g_{\mu\tau}g_{\lambda\nu}) \\
& + g_{\sigma\mu}(g_{\lambda\nu}g_{\rho\tau} - g_{\lambda\tau}g_{\nu\rho}) + g_{\sigma\nu}(g_{\mu\tau}g_{\rho\lambda} - g_{\mu\lambda}g_{\rho\tau})\big] = 0.
\end{aligned}
$$

We have obtained (158) by combining (154) with (134), and (159) by combining (154) with (155).

The law of gravitation chosen by Einstein is (154). These laws will be discussed in the next chapter.

CHAPTER VII

APPLICATIONS TO THE SOLAR FIELD

In the preceding chapter we saw that the only possible laws of gravitation under our postulates are given by (94), (154), (134), and (155–159). The law (94) can be omitted, as it determines but one of the ten $g_{\mu\nu}$. Equation (154) is Einstein's law, which will be considered later.†

The equation (158) can not be used as a law of gravitation. For in case the total curvature G is zero it becomes $R_{\rho\mu\nu\sigma}=0$, a necessary and sufficient condition that space-time be flat (§22). Thus the field is reducible by a transformation of coördinates to that of the Special Theory of Relativity, which by definition means the absence of all matter. If G is not zero, it is in any case a constant. Similarly in this case it can be shown‡ that all solutions are deducible from any one of them by a transformation of coördinates. On this basis we shall reject (158).

We shall now consider the equation

$$
(134)\qquad W_{\rho\mu\nu\sigma} \equiv R_{\rho\mu\nu\sigma} + \tfrac{1}{2}G_{\rho\nu}g_{\mu\sigma} - \tfrac{1}{2}G_{\rho\sigma}g_{\mu\nu} - \tfrac{1}{2}G_{\mu\nu}g_{\rho\sigma} + \tfrac{1}{2}G_{\mu\sigma}g_{\rho\nu} + \tfrac{1}{6}G(g_{\rho\sigma}g_{\mu\nu} - g_{\rho\nu}g_{\mu\sigma}) = 0.
$$

It has been shown†† that this equation represents a necessary and sufficient condition that ds^2 be reducible to the form

$$
(160)\qquad ds^2 = h\left[-\sum_{i=1}^{3}dx_i^2 + dx_4^2\right],
$$

where h is a function of the x_i. The fact that the condition is necessary indicates that (160) is a solution of (134) regardless of the form of the function h. It is clear from this remark how indeterminate a theory we should obtain if we used (134) as a law of gravitation. In a given physical problem no integration of a set of partial differential equations would be required; it would merely be necessary to assign to the function h any value satisfying the given boundary conditions. This does not mean, however, that the motion of a planet in the field of the sun, to take a physical example, is arbitrary under the law (134). We shall show later what limitations to the motion are imposed by the form of the solution (160), and in particular that what might be called a "pseudo-Newtonian" situation is admissible, in which a planet revolves around the sun with its perihelion

† When the text of this chapter was written we had not as yet found the tensor $S_{\sigma\tau\rho\mu\nu\lambda}$. Rather than revise this chapter we have inserted a few remarks on the subject in Appendix VI.

‡ Veblen: Invariants of Quadratic Differential Forms, p. 78.

†† J. A. Schouten: *Math. Zeitschr.*, 11, 1921, p. 58.

point fixed in heliocentric coördinates. Furthermore, it follows at once from the form of (160) that a ray of light is not deflected in passing the sun. We obtain, then, a gravitational field in which the path of a planet differs inappreciably from that required by the Newtonian law, and in which a ray of light behaves as in the classical electro-magnetic theory, as distinguished from the corpuscular theory. The fact that such a result is permissible under the fundamental postulates of Relativity, as set forth in §34, is interesting.

It is to be especially noted that (134), contrary to Einstein's law (154), permits the total curvature G to be other than a constant. It will be shown later that in the field of a mass-point, G is a monotonic decreasing function of the radius r, and at infinity approaches a constant given by the boundary conditions. Thus under this law every particle of matter is related to a variation in the total curvature, which approaches zero as the distance from the particle increases, and may be thought of as superposed on the constant total curvature of the universe. On this theory it would be natural to expect the total curvature to be sensibly increased at a point inside a region containing a large number of particles, i.e., inside of "solid matter." This type of gravitational theory seems to us a much more natural one than that arising from the usual Relativity law, since the latter calls for a sudden jump in the total curvature at the boundary of a mass. In order to obtain such a theory, and at the same time admit the Einstein deflection of light, it would be necessary to renounce one or both of Postulates IV and V of §34.†

We can combine equations (134) and (94). Then the function h of (160) ceases to be arbitrary. In particular we shall consider the case $G=0$. Then

$$R_{\rho\mu\nu\sigma}+\tfrac{1}{2}G_{\rho\nu}g_{\mu\sigma}-\tfrac{1}{2}G_{\rho\sigma}g_{\mu\nu}-\tfrac{1}{2}G_{\mu\nu}g_{\rho\sigma}+\tfrac{1}{2}G_{\mu\sigma}g_{\rho\nu}=0. \tag{161}$$

We require a solution of this set of equations for the solar field. We also wish to know the form of the solution (160) of (134) in another system of coördinates; instead of transforming (160) we shall set up the required equations directly in the new system.

It has been shown‡ that in polar coördinates the ds^2 for a static, radially symmetric field may be written

$$ds^2=-e^{\lambda}dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2+e^{\nu}dt^2, \tag{162}$$

where λ and ν are functions of r alone, such that as $r\mp\infty$, $e^{\lambda}\doteq e^{\nu}\doteq 1$, using light units. For this ds^2 all equations of (134) are identically zero except those containing exactly two distinct indices. Of these six equations only the following two are distinct:

† Or we could adopt (155) as the law of gravitation. Cf. Appendix VI.

‡ H. Weyl: *Raum. Zeit. Materie.* §31.

$$(163)\qquad \begin{aligned} \nu'' - \frac{1}{2}\lambda'\nu' + \frac{1}{2}\nu'^2 + \frac{1}{r}\nu' - \frac{1}{r}\lambda' - 2\frac{1}{r^2}e^{\lambda} + 2\frac{1}{r^2} + \frac{2}{3}e^{\lambda}G = 0, \\ \frac{1}{r}(\nu' - \lambda') + \frac{1}{3}e^{\lambda}G = 0. \end{aligned}$$

We also find that

$$G = -e^{-\lambda}\left[\nu'' - \frac{1}{2}\lambda'\nu' + \frac{1}{2}\nu'^2 + 2\frac{1}{r}\nu' - 2\frac{1}{r}\lambda' - 2\frac{1}{r^2}e^{\lambda} + 2\frac{1}{r^2}\right].$$

If this value of G be substituted in (163) both equations reduce to

$$(164)\qquad \nu'' - \frac{1}{2}\lambda'\nu' + \frac{1}{2}\nu'^2 + \frac{1}{r}\lambda' - \frac{1}{r}\nu' - 2\frac{1}{r^2}e^{\lambda} + 2\frac{1}{r^2} = 0,$$

as is required by the fact that h in (160) is arbitrary. Thus we may choose ν arbitrarily, subject to the boundary conditions,† and λ will be determined by (164). Put $\lambda = \log \lambda_1$ and $\nu = \log \nu_1$, and assume that λ_1 and ν_1 are expansible as convergent power series in m/r, so that

$$(165)\qquad \begin{aligned} \lambda_1 &= 1 + \frac{a_1 m}{r} + \frac{a_2 m^2}{r^2} + \cdots \\ \nu_1 &= 1 + \frac{b_1 m}{r} + \frac{b_2 m^2}{r^2} + \cdots. \end{aligned}$$

Then it is found by substituting in (164) that $a_1 = b_1$, $a_2 = 2b_2 - \frac{3}{4}b_1^2, \cdots$. Then

$$G = \left(\frac{9}{2}b_1^2 - 6b_2\right)\frac{m^2}{r^4} + \cdots.$$

We shall now consider the case where $G = 0$. Then the second equation of (163) gives $\lambda' = \nu'$, or $\lambda = \nu$, by the boundary conditions. The first of (163) becomes

$$\nu'' - \frac{2}{r^2}e^{\nu} + \frac{2}{r^2} = 0,$$

which admits the solution

$$\nu = \log\left(1 + \frac{b_1 m}{r} + \frac{3}{4}\frac{b_1^2 m^2}{r^2} + \cdots\right),$$

where b_1 is an arbitrary constant.

Thus in (162)

$$(166)\qquad -g_{11} = g_{44} = 1 + \frac{b_1 m}{r} + \frac{3}{4}\frac{b_1^2 m^2}{r^2} + \cdots.$$

† We shall confine ourselves to the case where one boundary condition states that $\lambda = \nu = 0$ for $r = \infty$, and that λ and ν vanish with m. In this case (154) admits no solutions except those of the equations $G_{\mu\nu} = 0$.

The Schwartzschild solution of $G_{\mu\nu}=0$ is

$$\nu_1 = \frac{1}{\lambda_1} = 1 + \frac{b_1 m}{r}\,.$$

The constant b_1 is undetermined.

We shall now study the solutions of the equations of motion of a mass-point in a radially symmetric field of the type we have been discussing. The equations of motion are those of the geodesic lines:

$$\frac{d^2 x_i}{ds^2} + \sum_{\alpha,\beta} \begin{Bmatrix} \alpha\beta \\ i \end{Bmatrix} \frac{dx_\alpha}{ds}\frac{dx_\beta}{ds} = 0. \tag{167}$$

Since the form of (162) is independent of the choice of gravitational law, we obtain great generality by avoiding the introduction of numerical quantities until after the integration. We shall merely assume that the $g_{\mu\nu}$ of (162) are analytic functions of the variable m/r in every finite domain, so that

$$\begin{aligned} e^\lambda &= \lambda_1 = 1 + \frac{a_1 m}{r} + \frac{a_2 m^2}{r^2} + \cdots \\ e^\nu &= \nu_1 = 1 + \frac{b_1 m}{r} + \frac{b_2 m^2}{r^2} + \cdots\,. \end{aligned} \tag{168}$$

Then the equations (167) become

$$\frac{d^2r}{ds^2} + \frac{1}{2}\frac{1}{\lambda_1}\frac{d\lambda_1}{dr}\left(\frac{dr}{ds}\right)^2 - \frac{r}{\lambda_1}\left(\frac{d\theta}{ds}\right)^2 - \frac{r\sin^2\varphi}{\lambda_1}\left(\frac{d\phi}{ds}\right)^2 + \frac{1}{2}\frac{1}{\lambda_1}\frac{d\nu_1}{dr}\left(\frac{dt}{ds}\right)^2 = 0$$

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} = 0$$

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} + 2\cot\varphi\frac{d\theta}{ds}\frac{d\phi}{ds} = 0$$

$$\frac{d^2t}{ds^2} + \frac{1}{2}\frac{1}{\nu_1}\frac{d\nu_1}{dr}\left(\frac{dt}{ds}\right)^2 = 0\,.$$

If $\phi=\pi/2$ and $d\phi/ds=0$ initially, the third equation shows that the orbit remains in the plane $\phi=\pi/2$. The second and fourth equations lead respectively to the integrals

$$\begin{aligned} r^2\frac{d\theta}{ds} &= h \\ \frac{dt}{ds} &= \frac{k}{\nu_1}, \end{aligned} \tag{169}$$

where h and k are constants of integration. Using the relation

$$ds^2 = -\lambda_1 dr^2 - r^2 d\theta^2 + \nu_1 dt^2 \tag{170}$$

with (169), and putting $u = 1/r$, we are lead to the equation of the orbit

$$\left(\frac{du}{d\theta}\right)^2 = \frac{k^2}{h^2}\frac{1}{\lambda_1\nu_1} - \frac{1}{h^2}\frac{1}{\lambda_1} - \frac{u^2}{\lambda_1}. \tag{171}$$

We shall put $\alpha^2 = 1/h^2$, $\beta^2 = k^2/h^2$. Then by (168) the right side is expressible as a power series in u, α^2, β^2, and (171) becomes

$$\begin{aligned}\left(\frac{du}{d\theta}\right)^2 = \frac{1}{\lambda_1}\left[\frac{\beta^2}{\nu_1} - \alpha^2 - u^2\right] &= -\alpha^2 + \beta^2 + ma_1\alpha^2 u - me_1\beta^2 u \\ &- u^2 + m^2e_2\alpha^2u^2 + m^2f_2\beta^2u^2 + ma_1u^3 \\ &+ m^3e_3\alpha^2u^3 + \cdots = P(u),\end{aligned} \tag{172}$$

where

$$\begin{aligned} e_1 &= a_1 + b_1 \\ e_2 &= a_2 - a_1^2 \\ f_2 &= a_1^2 - a_2 + b_1^2 - b_2 + a_1b_1 \\ e_3 &= a_1^3 - 2a_1a_2 + a_3. \end{aligned}$$

The unwritten terms are of order four, at the least, in u. If we differentiate (172) we obtain the equation

$$\begin{aligned}\frac{d^2u}{d\theta^2} = \frac{1}{2}ma_1\alpha^2 - \frac{1}{2}me_1\beta^2 - u + m^2e_2\alpha^2u + m^2f_2\beta^2u \\ + \frac{3}{2}ma_1u^2 + \frac{3}{2}m^3e_3\alpha^2u^2 + \cdots = Q(u).\end{aligned} \tag{173}$$

We shall show that there exists a circular orbit, i.e., a solution $u =$ constant of (172), which also satisfies (173), so that $P(u) = 0$ and $Q(u) = 0$. Now the functional determinant of P and Q with respect to u and β^2 does not vanish for $u = \beta^2 = \alpha^2 = 0$. Therefore by a well-known theorem† on implicit functions these equations admit a unique solution for u and β^2 as power series in α^2, vanishing with α^2, and convergent for $|\alpha^2| <$ some positive constant. The solution is

$$\left\{\begin{aligned} u &= \alpha^2 A = \alpha^2\left[-\frac{1}{2}mb_1 + m^3\left(\frac{1}{2}b_1b_2 - \frac{3}{8}b_1^3\right)\alpha^2 + \cdots\right] \\ \beta^2 &= \alpha^2 B = \alpha^2\left[1 - \frac{1}{4}m^2b_1^2\alpha^2 + \cdots\right]. \end{aligned}\right. \tag{174}$$

A and B are defined by these equations. The second gives

$$k^2 = 1 - \frac{1}{4}m^2b_1^2\alpha^2 + \cdots. \tag{175}$$

† Goursat-Hedrick: Mathematical Analysis. Vol. I, Chap. IX, §187. F. R. Moulton: Periodic Orbits. Chap. I, §1.

These values for u and β^2 define, then, circular orbits such that $P=Q=0$. We note that if α^2 is sufficiently small $k^2<1$, i.e., $\alpha^2>\beta^2$, and $b_1<0$, provided that the orbits are real.

We wish to study a particular class of the solutions of (172) or of (173), those that differ but slightly from the circular orbits just defined. We shall later impose the further condition that the orbits be closed with respect to a new independent variable, a constant multiple of θ. We put

$$\beta^2 = \alpha^2(B + A^2\alpha^2 de), \tag{176}$$

where d and e are new constants, and make the change of variable

$$u = \alpha^2 A(1 + e\rho). \tag{177}$$

We impose the initial conditions for $\theta=0$, $u(0)\equiv U=\alpha^2A(1+e)$, or $\rho(0)=1$ and further

$$\frac{du}{d\theta} = \frac{d\rho}{d\theta} = 0 \text{ for } \theta = 0.$$

Thus for $\theta = 0$

$$U = \alpha^2 A(1 + e). \tag{178}$$

Substituting for A its value from (174), this equation becomes

$$U = (1 + e)[-\tfrac{1}{2}mb_1\alpha^2 + \alpha^4(\text{other terms})].$$

If $|e|<1$ and α^2 is sufficiently small, this requires $b_1<0$, as in the circular case, since U is positive by definition.

Before proceeding to the solution of (173) we shall show that the constants of integration h and k, introduced in (169), can be expressed as power series in U and e, and conversely that e is expressible in terms of h and k, so that the equation of the orbit (177) contains only these constants. It will be seen that the constant d of (176) is introduced merely for convenience, and is directly expressible as a power series in α^2 and e.

Now (178) can be solved at once for α^2 as a unique power series in U and e, of the form

$$\alpha^2 = U\left(-\frac{2}{mb_1} + \frac{2}{mb_1}e + \cdots\right).$$

To obtain the corresponding value of k we substitute (176) and (178) in (172) for $\theta=0$, i.e., in $P=0$, remembering that $u=\alpha^2A$, $\beta^2=\alpha^2B$ satisfy $P=0$ and $Q=0$. The calculation is simplified by use of the following lemma:

Consider a power series

$$P_1(u) = p_0 + p_1u + p_2u^2 + \cdots = 0,$$

with a root $u=A$. If we put $u=A(1+e)$, we find

$$P_1 = p_1Ae + 2p_2A^2e + p_2A^2e^2 + \cdots = 0.$$

From the Binomial Theorem it follows that the terms in e may be written

$$e \sum_n n p_n A^n.$$

Now suppose that $u = A$ is also a root of the derived equation $P_1'(u) = 0$. Then

$$P_1'(u) = p_1 + 2p_2 A + 3p_3 A^2 + \cdots = 0.$$

If this equation be multiplied by Ae it may be written

$$e \sum_n n p_n A^n = 0.$$

Thus P_1 contains no terms in e^1.

Using this lemma the result of our substitution in (172) is, after dividing by $A^2\alpha^4 e$,

$$d - me_1 A\alpha^2 d - me_1 A\alpha^2 de - e + m^2 e_2 \alpha^2 e + m^2 f_2 A^2 \alpha^4 d + 2m^2 f_2 A^2 \alpha^4 de$$
$$+ m^2 f_2 A^2 \alpha^4 de^2 + m^2 f_2 B\alpha^2 e + 3ma_1 A\alpha^2 e + ma_1 A\alpha^2 e^2 + \cdots = 0.$$

The omitted terms contain α^4 as a factor, vanish for $d = e = 0$, and contain no terms of the form $d^0 e^1$. This equation can be solved for d uniquely as a power series in e and α^2, vanishing with e; the explicit solution is

$$d = e(1 + \alpha^2 D), \tag{179}$$

where

$$D = m^2(a_1 b_1 - \tfrac{1}{2} b_1^2 - (e_2 + f_2) - \tfrac{1}{2} b_1^2 e) + \alpha^2(\text{other terms}).$$

If we substitute this result in (176) we obtain

$$\beta^2 = \alpha^2 B + \alpha^4 A^2 e^2 (1 + \alpha^2 D),$$

or

$$k^2 = 1 - \tfrac{1}{4} m^2 b_1^2 \alpha^2 (1 - e^2) + \alpha^4(\text{other terms}) \tag{180}$$

It is clear from this that if α^2 is small, and if $e^2 < 1$, then $k^2 < 1$, or $\alpha^2 > \beta^2$. As e^2 approaches zero k^2 approaches the value B of (174), which leads to a circular orbit. This is clearly the minimum value of k^2 for all values of e that satisfy the condition necessary to secure the convergence of (179). Thus k^2, and hence k, is determined in terms of α^2, e, or, by a previous remark, in terms of U, e.

In integrating (173) two new constants must be introduced. One of them is known at once, since (173) was derived from (172), so that the latter serves as an integral with known constant. The second new constant of integration, which enters as an additive constant to θ, was taken to be zero when we took $\theta = 0$ as the value of θ corresponding to the nearest apse. It follows that the equation of the orbit (177) must be expressible as

a function of only $\alpha(=1/h)$, k, and θ. To accomplish this it is necessary to solve (180) for e. The equation may be written

$$(181) \qquad \left[1-\frac{4}{m^2b_1^2}\frac{\gamma}{\alpha^2}+\alpha^2B_1\right]^{1/2} = \pm\, e(1+\alpha^2D_1)^{1/2},$$

where $\gamma=1-k^2$, B_1 is a power series in α^2, and D_1 a power series in α^2, e. The left member, which is real, can be expanded as a convergent power series in γ/α^2 and α^2. As we wish e to be positive, the plus sign is required. We may then solve for e as a power series in γ/α^2 and α^2, of the form

$$e = 1 - E\left(\frac{\gamma}{\alpha^2},\ \alpha^2\right),$$

where E is a power series in γ/α^2 and α^2, and $1>E>0$. The series will converge for sufficiently small values of γ/α^2 and α^2. The form of (180) shows that $4\gamma/m^2b_1^2\alpha^2$ is very slightly less than unity for small values of e^2, so that the left member of (181) is small. We have the inequality $\gamma/\alpha^2<\frac{1}{4}m^2b_1^2+\alpha^2$ (other terms), or approximately, $\gamma/\alpha^2<\frac{1}{4}m^2b_1^2$.

We turn now to the solution of (173), making the change of independent variable

$$(182) \qquad \theta = (1+\delta)^{1/2}\psi,$$

where δ is a constant. It will be proved that there exists a value of δ such that (173) admits as solution a two-parameter family of closed curves, reducing to circles for $e=0$; in other words there exist periodic solutions of period 2π, for small values of e.

Now substitute (176), (177), and (182) in (173), and eliminate d by (179); there results

$$\begin{aligned}(183)\quad \frac{d^2\rho}{d\psi^2}+(1+\delta)\rho = (1+\delta)\Big\{ & m^2\left[e_2+f_2-\frac{3}{2}a_1b_1\right]\alpha^2\rho+\frac{1}{4}m^2b_1e_1\alpha^2e \\ & -\frac{3}{4}m^2a_1b_1\alpha^2e\rho^2 - m^4\left[\frac{1}{4}b_1^2f_2+\frac{3}{2}b_1e_3-\frac{3}{2}a_1b_1b_2\right. \\ & \left.+\frac{9}{8}a_1b_1^3\right]\alpha^4\rho + m^4\left[\frac{1}{4}a_1b_1^2e_1-\frac{1}{8}b_1^3e_1-\frac{1}{4}b_1e_1(e_2+f_2)\right. \\ & \left.+\frac{1}{4}b_1^2f_2-\frac{1}{4}b_1e_1b_2+\frac{3}{16}b_1^3e_1\right]\alpha^4e + m^4\left[\frac{3}{4}a_1b_1b_2-\frac{9}{16}a_1b_1^3\right. \\ & \left.-\frac{3}{4}b_1e_3\right]\alpha^4e\rho^2-\frac{1}{8}m^4b_1^3e_1\alpha^4e^2+\frac{1}{4}m^4b_1^2f_2\alpha^4e^2\rho \\ & +\alpha^6(\text{other terms}).\end{aligned}$$

Sufficient conditions that a solution of an equation of this form be periodic are

$$\rho(2\pi) = \rho(0)$$
$$\rho'(2\pi) = \rho'(0).$$

In this case these equations are not independent,† the second implying the first, due to the presence of (172). It will be sufficient, then, to determine δ such that $\rho'(2\pi)=0$.

By the Cauchy-Poincaré theorem equation (183) can be integrated as a power series for ρ in α^2, e, convergent for $0 \leqq \psi \leqq 2\pi$, provided that α^2 and e are sufficiently small. The initial conditions are now written in the form

(184) $$\rho(0) \underset{\alpha^2, e}{\equiv} 1 ; \qquad \rho'(0) \underset{\alpha^2, e}{\equiv} 0.$$

If this integration is carried out, we find that the equation $\rho'(2\pi)=0$ becomes

(185) $$\rho'(2\pi) = -\pi\delta + \pi\left[m^2(e_2 + f_2) - \frac{3}{2}m^2 a_1 b_1\right]\alpha^2 + P(\alpha^2, e, \delta) = 0,$$

where $P(\alpha^2, e, \delta)$ is a power series in the quantities indicated. It vanishes for $\alpha^2=\delta=0$, thus containing no terms in e alone. For if $\alpha^2=0$ in (183) there is a solution of period 2π if and only if $\delta=0$. Thus (185), for $\alpha^2=0$, must contain the factor δ. It follows that (185) admits a unique solution for δ as a power series in α^2 and e, vanishing with α^2, and convergent for α^2 and e sufficiently small, which we shall write

(186) $$\delta = \delta_{20}\alpha^2 + \delta_{21}\alpha^2 e + \delta_{40}\alpha^4 + \delta_{41}\alpha^4 e + \delta_{42}\alpha^4 e^2 + \alpha^6(\text{other terms}).$$

This expansion contains no terms in $\alpha^n e^n$, $\alpha^2 e$, nor in $\alpha^4 e^3$, for no such terms are present in (183), and therefore in (185). With this value for δ equation (183) admits a periodic solution of period 2π. The coefficients δ_{ij} can be computed from (185). We shall, however, obtain the same result in the direct construction of the solution.‡

We can now integrate (183), after substituting (186) in it, in the form

(187) $$\rho = \rho_{00} + \rho_{20}\alpha^2 + \rho_{21}\alpha^2 e + \rho_{40}\alpha^4 + \rho_{41}\alpha^4 e + \rho_{42}\alpha^4 e^2 + \alpha^6(\text{other terms}),$$

where the ρ_{ij} are each periodic in ψ with period 2π, and are subject also to the initial conditions (184). Substituting, then, (186) and (187) in (183), we obtain the set of equations

† Cf. Moulton: loc. cit. Chap. II, §40. Poincaré: *Méthodes Nouvelles de la Mécanique Céleste.* Vol. I p. 87.

‡ The existence of the periodic solution also follows from a general existence theorem due to W. D. MacMillan: Transactions of the Amer. Math. Society vol. XIII, 1912, p. 157.

$$(188)\quad (a)\quad \rho_{00}'' + \rho_{00} = 0$$

$$(b)\quad \rho_{20}'' + \rho_{20} = -\delta_{20}\rho_{00} + [m^2(e_2 + f_2) - \tfrac{3}{2}m^2 a_1 b_1]\rho_{00}$$

$$(c)\quad \rho_{21}'' + \rho_{21} = -\delta_{21}\rho_{00} + \tfrac{1}{4}m^2 b_1 e_1 - \tfrac{3}{4}m^2 a_1 b_1 \rho_{00}^2 .$$

The writing down of further equations is simplified by use of the solutions of these. In integrating each equation two constants of integration enter; they are uniquely determined by the initial conditions (184). The δ_{ij} are uniquely determined by the periodicity conditions, as previously noted. In order that an equation of the type of (188) admit a solution of period 2π, it is necessary that the coefficient of each term in $\cos\psi$ be identically zero. Now the required solution of (188a) is $\rho_{00} = \cos\psi$, and since the equation for the general ρ_{ij} contains precisely one term in δ_{ij}, which is of the form $\delta_{ij}\rho_{00}$, it is clear that the above condition uniquely determines the δ_{ij}, for all values of i and j. Furthermore, the right members of equations (188) are expressible in powers of $\cos\psi$, that is to say in cosines of integral multiples of ψ. None of these terms can give rise to non-periodic terms except those in $\cos\psi$, and these latter will have zero coefficients.

The solutions of the first three equations of (188) are

$$(189)\quad (a)\quad \rho_{00} = \cos\psi$$

$$(b)\quad \rho_{20} = 0$$

$$(c)\quad \rho_{21} = m^2\{\tfrac{1}{4}b_1^2 - \tfrac{1}{8}a_1 b_1 - \tfrac{1}{4}b_1^2\cos\psi + \tfrac{1}{8}a_1 b_1 \cos 2\psi\},$$

and

$$(190)\quad (b)\quad \delta_{20} = m^2\{e_2 + f_2 - \tfrac{3}{2}a_1 b_1\}$$

$$(c)\quad \delta_{21} = 0 .$$

We can now simplify the further equations of (188), obtaining

$$(188)\quad (d)\quad \rho_{40}'' + \rho_{40} = \cos\psi\{-\delta_{40} + \delta_{20}^2 + \tfrac{1}{2}m^4[\tfrac{9}{4}a_1 b_1^3 - 3a_1 b_1 b_2 + 3b_1 e_3 - \tfrac{1}{2}b_1^2 f_2]\}$$

$$\begin{aligned}(e)\quad \rho_{41}'' + \rho_{41} = {} & -\delta_{41}\cos\psi + \tfrac{1}{2}m^2[\tfrac{1}{2}b_1 e_1 - 3a_1 b_1]\delta_{20} \\ & + \tfrac{1}{4}m^4[a_1^2 b_1^2 + \tfrac{1}{8}a_1 b_1^3 + \tfrac{1}{4}b_1^4 + \tfrac{1}{2}a_1 b_1 b_2 \\ & - b_1^2 b_2 - b_1^2 e_2 - \tfrac{1}{4}a_1 b_1(e_2 + f_2)] \\ & + \cos 2\psi\{-\tfrac{3}{8}m^2 a_1 b_1 \delta_{20} + \tfrac{1}{8}m^4[3a_1 b_1 b_2 - \tfrac{9}{4}a_1 l_1 \\ & - 3b_1 e_3]\}\end{aligned}$$

$$(f)\quad \rho_{42}'' + \rho_{42} = \cos\psi\{-\delta_{42} + \tfrac{1}{4}m^4 b_1^2 f_2 - \tfrac{3}{2}m^2 a_1 b_1 \rho_{21}\} - \tfrac{1}{8}m^4 b_1^3 e_1 .$$

$\vdots$

The solutions are

$$(189)\quad (d)\ \rho_{40} = 0$$

$$(e)\ \rho_{41} = \tfrac{1}{4}m^4\{\tfrac{7}{4}a_1^2b_1^2 - \tfrac{11}{8}a_1b_1^3 + \tfrac{1}{4}b_1^4 + \tfrac{1}{2}a_1b_1b_2 - b_1^2b_2 + b_1^2f_2 - \tfrac{3}{2}b_1e_3 - \tfrac{3}{2}a_1b_1(e_2+f_2) + \cos\psi[-a_1^2b_1^2 + a_1b_1^3 - \tfrac{1}{4}b_1^4 + b_1e_3 - b_1^2f_2 + b_1^2b_2 + a_1b_1(e_2+f_2)] + \cos 2\psi[-\tfrac{3}{4}a_1^2b_1^2 + \tfrac{3}{8}a_1b_1^3 - \tfrac{1}{2}a_1b_1b_2 + \tfrac{1}{2}b_1e_3 + \tfrac{1}{2}a_1b_1(e_2+f_2)]\}$$

$$(f)\ \rho_{42} = \tfrac{1}{8}m^4\{\tfrac{1}{2}a_1b_1^3 - b_1^4 - \cos\psi[\tfrac{3}{32}a_1^2b_1^2 - b_1^4] - \tfrac{1}{2}a_1b_1^3\cos 2\psi + \tfrac{3}{32}a_1^2b_1^2\cos 3\psi\}$$

$$\vdots$$

and

$$(190)\quad (d)\ \delta_{40} = m^4\{\tfrac{9}{4}a_1^2b_1^2 - \tfrac{9}{8}a_1b_1^3 + \tfrac{3}{2}a_1b_1b_2 - \tfrac{3}{2}b_1e_3 - \tfrac{1}{4}b_1^2f_2 + (e_2+f_2)^2 - 3a_1b_1(e_2+f_2)\}$$

$$(e)\ \delta_{41} = 0$$

$$(f)\ \delta_{42} = \tfrac{1}{8}m^4\{\tfrac{3}{4}a_1^2b_1^2 - 3a_1b_1^3 + 2b_1^2f_2\}.$$

With these results the equation of the orbit (177) becomes, after returning to the variable $r = 1/u$,

$$(191)\qquad r = \frac{\dfrac{1}{\alpha^2A}}{1 + e\rho},$$

where

$$(192)\quad \rho = \cos\psi + \tfrac{1}{4}m^2\{b_1^2 - \tfrac{1}{2}a_1b_1 - b_1^2\cos\psi + \tfrac{1}{2}a_1b_1\cos 2\psi\}\alpha^2e + \tfrac{1}{4}m^4\{\tfrac{5}{4}a_1^2b_1^2 - \tfrac{15}{8}a_1b_1^3 - \tfrac{3}{2}a_1^3b_1 + \tfrac{5}{4}b_1^4 + 3a_1b_1a_2 + 2a_1b_1b_2 - b_1^2a_2 - 2b_1^2b_2 - \tfrac{3}{2}b_1a_3 + \cos\psi[-a_1^2b_1^2 + a_1b_1^3 + a_1^3b_1 - \tfrac{5}{4}b_1^4 - 2a_1b_1a_2 - a_1b_1b_2 + b_1^2a_2 + 2b_1^2b_2 + b_1a_3] + \cos 2\psi[-\tfrac{1}{4}a_1^2b_1^2 + \tfrac{7}{8}a_1b_1^3 + \tfrac{1}{2}a_1^3b_1 - a_1b_1a_2 - a_1b_1b_2 + \tfrac{1}{2}b_1a_3]\}\alpha^4e + \tfrac{1}{4}m^4\{\tfrac{1}{4}a_1b_1^3 - \tfrac{1}{2}b_1^4 + \cos\psi[\tfrac{1}{2}b_1^4 - \tfrac{3}{64}a_1^2b_1^2] - \tfrac{1}{4}a_1b_1^3\cos 2\psi + \tfrac{3}{64}a_1^2b_1^2\cos 3\psi\}\alpha^4e^2 + \cdots.$$

We have replaced e_1, e_2, f_2, and e_3 by their values as given by (172).

If the perihelion distance, equal to $1/U$, is as great as in the case of the planet Mercury, the constant α^2 is extremely small, and the orbit will differ but slightly from a fixed ellipse in the ψ-space, which is rotating uniformly with respect to the θ-space. The constant e is the measured eccentricity of the planet's orbit; it is related to the constants of integration of our problem by (181). It is most convenient to introduce the major semi-axis a, instead of using the determination of α^2 in terms of U. Then, considering the orbit as an ellipse in the ψ-space, we have

$$\frac{1}{\alpha^2 A} = a(1 - e^2),$$

or

$$\alpha^2 = -\frac{2}{mb_1(1 - e^2)a}\left[1 - m^2\left(\frac{3}{4}b_1^2 - b_2\right)\alpha^2 + \cdots\right].$$

This equation can be solved uniquely for α^2 as a power series in $1/a$, giving

$$(193)\quad \alpha^2 = -\frac{2}{mb_1(1 - e^2)}\cdot\frac{1}{a} + \left[\frac{4b_2}{b_1^2(1-e^2)^2} - \frac{3}{(1-e^2)^2}\right]\frac{1}{a^2} + \cdots .$$

Since $\theta = \sqrt{1+\delta}\psi$, and δ is given by (186) and (190), we obtain at once a formula for the angular change in the perihelion point per revolution, measured in radians. Denoting this change by Δ we find

$$\begin{aligned}(194)\quad \Delta = 2\pi(1 + \delta)^{1/2} - 2\pi &= \pi m^2[b_1^2 - b_2 - \tfrac{1}{2}a_1b_1]\alpha^2 \\ &+ \pi m^4[\tfrac{1}{2}b_1^4 - \tfrac{1}{16}a_1^2b_1^2 - \tfrac{17}{8}a_1b_1^3 - \tfrac{3}{2}a_1^3b_1 - \tfrac{5}{4}b_1^2b_2 + \tfrac{9}{4}a_1b_1b_2 \\ &+ 3a_1b_1a_2 + \tfrac{1}{4}b_1^2a_2 - \tfrac{3}{2}b_1a_3 + \tfrac{3}{4}b_2^2]\alpha^4 \\ &+ \pi m^4[\tfrac{1}{4}b_1^4 + \tfrac{11}{32}a_1^2b_1^2 - \tfrac{1}{8}a_1b_1^3 - \tfrac{1}{4}b_1^2a_2 - \tfrac{1}{4}b_1^2b_2]\alpha^4e^2 + \cdots .\end{aligned}$$

In this result we eliminate α^2 by use of (193), and Δ is expressed in terms of only a_i, b_i, a and e; a positive value corresponds to an advance of perihelion in the direction of the planet's motion. For our present purposes we can neglect all but the first term, giving

$$(195)\qquad \Delta = -\frac{2\pi m}{(1 - e^2)a}\left[b_1 - \frac{b_2}{b_1} - \frac{1}{2}a_1\right].$$

It is of interest to note that of the coefficients a_i, b_i, this result contains only a_1, b_1 and b_2. Under our initial assumption concerning the singularities of the $g_{\mu\nu}$ in this problem, our formulae for Δ are valid for any law of gravitation satisfying the first three postulates of §34. In other words the validity of the formulae does not depend on the linearity of the equation expressing the law in the second derivatives, nor on the order of derivatives that enter.

For Einstein's law $G_{\mu\nu}=0$, we obtain from the Schwarzschild solution, (p. 91), taking $b_1=-2$,

$$b_1 = -a_1 = -2, \quad b_2 = 0.$$

Then

$$\Delta = \frac{6\pi m}{(1-e^2)a}. \tag{196}$$

This is the well-known result, leading, in the case of the planet Mercury, to an advance of perihelion of approximately 43″ per century.

Now consider the law (161), which is equivalent to (134) with $G=0$. Its solution for this case is given in (166). Giving the arbitrary constant b_1 the value -2, as in the Einstein case, we have

$$b_1 = a_1 = -2, \quad b_2 = 3.$$

Then

$$\Delta = -\frac{\pi m}{(1-e^2)a}. \tag{197}$$

Thus the law (161) calls for a retrograde motion of perihelion at the rate of one sixth that of the Einstein law, or roughly $-7''$ per century.

In the case of the law (134) it will be recalled that the b_i of (165) may be arbitrarily chosen. As in previous cases we take $b_1=-2$; then $a_1=-2$. The formula (195) then becomes

$$\Delta = -\frac{2\pi m}{(1-e^2)a}\left[\frac{1}{2}b_2 - 1\right]; \tag{198}$$

here b_2 is entirely arbitrary. By assigning $b_2=-4$ we obtain exactly the advance of perihelion indicated in (196) for the Einstein law. For $b_2=2$ we have $\Delta=0$. Since the law (134) implies a zero deflection of light rays, this case represents precisely the state of affairs in the classical theory before the advent of Relativity. Its interest here is due to the fact that it is permissible under the postulates of §34. It does not seem, furthermore, that we can introduce any further postulates which are physically justifiable, and which lead to the elimination of the law (134). It is surely not justifiable to postulate that the divergence of the tensor be *identically* zero,† since we are interested only in the solutions of the tensor equation, and the divergence is always zero on the solutions (§ 45). A further discussion of the postulates involves the introduction of a matter tensor. The argument then depends on the concept of continuous regions of matter. But there seems no justification for introducing postulates that cannot be

† It is worthy of note that of the possible laws of gravitation listed in §45, only $S^{\lambda}{}_{\sigma\tau\rho\mu\nu}=0$ has its divergence identically zero, by (151).

based on a consideration of the gravitational field in free space alone. In any event, we shall limit our attention in this treatment to the latter case.

Knowing the equation of the orbit of a planet in the sun's field, we can easily find its period of revolution. From the equations (169) and (170), substituting (182), we obtain

$$\frac{dt}{d\psi}=\frac{\beta(1+\delta)^{1/2}}{\nu_1}r^2. \tag{199}$$

The value of ν_1 is given in (168), that of r in (191) and (192), that of δ in (186) and (190), and β is given by (180). If we make all of these substitutions and arrange the result according to powers of α the equation becomes

$$\begin{aligned}\frac{dt}{d\psi}=&\frac{4}{m^2b_1^2}\frac{1}{(1+e\cos\psi)^2}\frac{1}{\alpha^3}+\frac{1}{(1+e\cos\psi)^2}\frac{1}{\alpha}\left[-4-\frac{a_1}{b_1}\right.\\&\left.+6\frac{b_2}{b_1^2}-\frac{1}{2}(1-e^2)\right]+\frac{2}{1+e\cos\psi}\frac{1}{\alpha}\\&+\frac{1}{(1+e\cos\psi)^3}\frac{1}{\alpha}e^2\left[-2+\frac{a_1}{b_1}+2\cos\psi-\frac{a_1}{b_1}\cos 2\psi\right]\\&+\alpha\,(\text{other terms}).\end{aligned}\tag{200}$$

The period of revolution, from perihelion to perihelion, which we shall denote by T, is equal to the right member of this equation, integrated between the limits $\psi=0$ and $\psi=2\pi$.

The integration can be performed directly, since

$$\int_0^{2\pi}\frac{d\psi}{1+e\cos\psi}=\frac{2\pi}{(1-e^2)^{1/2}},$$

$$\int_0^{2\pi}\frac{d\psi}{(1+e\cos\psi)^2}=\frac{2\pi}{(1-e^2)^{3/2}},$$

and the remaining integrals can be evaluated in terms of these. Or the fourth term can be integrated as a power series in e, since $|e|<1$. Performing the integration we obtain

$$\begin{aligned}T=&\frac{8\pi}{m^2b_1^2(1-e^2)^{3/2}}\frac{1}{\alpha^3}+\frac{1}{\alpha}\left\{\frac{2\pi}{(1-e^2)^{3/2}}\left[-4-\frac{a_1}{b_1}+6\frac{b_2}{b_1^2}\right]\right.\\&\left.+\frac{3\pi}{(1-e^2)^{1/2}}+2\pi\left[\left(-2+\frac{a_1}{b_1}\right)e^2-3e^3+\cdots\right]\right\}+\cdots.\end{aligned}\tag{201}$$

For Einstein's law $G_{\mu\nu}=0$, we obtain from the Schwarzschild solution, (p. 91), taking $b_1=-2$,

$$b_1 = -a_1 = -2, \quad b_2 = 0.$$

Then

$$\Delta = \frac{6\pi m}{(1-e^2)a}. \tag{196}$$

This is the well-known result, leading, in the case of the planet Mercury, to an advance of perihelion of approximately 43″ per century.

Now consider the law (161), which is equivalent to (134) with $G=0$. Its solution for this case is given in (166). Giving the arbitrary constant b_1 the value -2, as in the Einstein case, we have

$$b_1 = a_1 = -2, \quad b_2 = 3.$$

Then

$$\Delta = -\frac{\pi m}{(1-e^2)a}. \tag{197}$$

Thus the law (161) calls for a retrograde motion of perihelion at the rate of one sixth that of the Einstein law, or roughly $-7''$ per century.

In the case of the law (134) it will be recalled that the b_i of (165) may be arbitrarily chosen. As in previous cases we take $b_1=-2$; then $a_1=-2$. The formula (195) then becomes

$$\Delta = -\frac{2\pi m}{(1-e^2)a}\left[\frac{1}{2}b_2 - 1\right]; \tag{198}$$

here b_2 is entirely arbitrary. By assigning $b_2=-4$ we obtain exactly the advance of perihelion indicated in (196) for the Einstein law. For $b_2=2$ we have $\Delta=0$. Since the law (134) implies a zero deflection of light rays, this case represents precisely the state of affairs in the classical theory before the advent of Relativity. Its interest here is due to the fact that it is permissible under the postulates of §34. It does not seem, furthermore, that we can introduce any further postulates which are physically justifiable, and which lead to the elimination of the law (134). It is surely not justifiable to postulate that the divergence of the tensor be *identically* zero,† since we are interested only in the solutions of the tensor equation, and the divergence is always zero on the solutions (§ 45). A further discussion of the postulates involves the introduction of a matter tensor. The argument then depends on the concept of continuous regions of matter. But there seems no justification for introducing postulates that cannot be

† It is worthy of note that of the possible laws of gravitation listed in §45, only $S^{\lambda}{}_{\sigma\tau\rho\mu\nu}=0$ has its divergence identically zero, by (151).

Consider now from (173) the equation $Q(u)=0$, or

$$-\tfrac{1}{2}mb_1\alpha^2 - u + m^2(e_2+f_2)\alpha^2 u + \tfrac{3}{2}ma_1u^2 + \cdots = 0\,;$$

it admits the solution

$$u = -\tfrac{1}{2}mb_1\alpha^2 + \cdots = \alpha^2 V_1.$$

We now transform (173) by

$$u = \alpha^2(V_1 + \rho),$$

and impose the initial conditions

$$\rho(0) = U_1 - V_1,$$

$$\rho'(0) = 0.$$

Then, by the Cauchy-Poincaré theorem the resulting equation may be integrated as a power series in α^2, convergent for any pre-assigned interval of θ, provided that α^2 is sufficiently small. The solution is easily constructed and will be omitted here. It is characterized as quasi-parabolic by the fact that u is zero for $\theta=\pi$.

We come now to the quasi-hyperbolic type of orbit, for which $k^2>1$. We shall not stop to treat the class of these orbits that differs but slightly from the quasi-parabolic class, but pass on to the consideration of those for which k^2 is large. If we allow the mass m to approach zero these orbits approach straight lines, to which they are related in the way that elliptic orbits of small eccentricity are related to circles.

Returning to the equation of the orbit (172) we determine a value of u such that $P(u)=0$, which we select as the value of u for $\theta=0$. To this end we put $\alpha^2=\eta\beta^2$, where the quantity $\eta\equiv 1/k^2$ may be regarded as small. Then the equation to be satisfied is

$$\beta^2 - \eta\beta^2\nu_1 - u^2\nu_1 = 0,$$

or

$$\beta^2(1-\eta) - mb_1\eta\beta^2 u - m^2b_2\eta\beta^2u^2 - u^2 - mb_1u^3 + u^4 \text{ (other terms)} = 0.$$

Now put

$$u = \beta(1-\eta)^{1/2}(1+v), \tag{204}$$

and the above equation becomes, after dividing by β^2

$$\begin{aligned}
&- mb_1\eta\beta - mb_1\eta\beta v - 2(1-\eta)^{1/2}v - (1-\eta)^{1/2}v^2\\
&- m^2b_2\eta(1-\eta)\beta^2 - 2m^2b_2\eta(1-\eta)\beta^2 v - m^2b_2\eta(1-\eta)\beta^2v^2\\
&- mb_1(1-\eta)^{3/2}\beta - 3mb_1(1-\eta)^{3/2}\beta v - 3mb_1(1-\eta)^{3/2}\beta v^2\\
&- mb_1(1-\eta)^{3/2}\beta v^3 + \cdots = 0.
\end{aligned}$$

It is easily seen that m and β enter this equation only in the form $(m\beta)^n$. It can be solved as a power series for v in $m\beta$, vanishing with $m\beta$. We thus obtain, substituting the solution in (204),

$$(205)\quad u = U = (1-\eta)^{1/2}\beta - \tfrac{1}{2}mb_1(1-\eta)^{3/2}\beta^2 - \tfrac{1}{2}mb_1\eta\beta^2 + \cdots .$$

Turning now to the equation (173) we write it

$$(206)\quad \frac{d^2u}{d\theta^2} + u = -\frac{1}{2}ma_1(1-\eta)\beta^2 - \frac{1}{2}mb_1\beta^2 + m^2e_2\eta\beta^2u$$
$$+ m^2f_2\beta^2u + \frac{3}{2}ma_1u^2 + \cdots .$$

Under the initial conditions

$$u(0) = U$$
$$u'(0) = 0,$$

this equation can be integrated as a power series in m and β, convergent for any pre-assigned interval of θ, provided that m and β are sufficiently small. The result of this integration is

$$(207)\quad u = (1-\eta)^{1/2}\beta\cos\theta + \tfrac{1}{4}ma_1(1-\eta)\beta^2 - \tfrac{1}{2}mb_1\beta^2$$
$$+ \tfrac{1}{2}mb_1\beta^2\left[1-(1-\eta)^{3/2}-\eta\right]\cos\theta - \tfrac{1}{4}ma_1(1-\eta)\beta^2\cos 2\theta + \cdots .$$

This is the equation of the orbit that we were seeking. The right side is analytic in η.

If in (207) we set $m=0$ we have

$$(208)\quad u = (1-\eta)^{1/2}\beta\cos\theta = \frac{(k^2-1)^{1/2}}{h}\cos\theta,$$

the equation of a straight line. Now the case $m=0$ is precisely that of the Special Theory of Relativity, whence we obtain an interesting interpretation of the constant k. From the Special Theory we have, in light units,

$$\left(\frac{ds}{dt}\right)^2 = 1 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2,$$

or

$$\frac{ds}{dt} = \pm(1-V^2)^{1/2},$$

where V is the velocity of the particle, in absolute value always less than or equal to unity. From (169) we have, for $m=0$,

$$\frac{ds}{dt} = \frac{1}{k},$$

so that, for k positive

$$k = \frac{1}{(1 - V^2)^{1/2}} \cdot$$

Thus in the limiting case $m=0$ the quantity k increases with the velocity. It follows by continuity that for small values of m and for orbits that do not differ too much from (208), large values of k correspond to large velocities of the planet. It will be seen below that for the limiting case of motion at the speed of light k is infinite.

Let us now consider the path of a ray of light which passes the mass m at a distance $R=1/U$. We obtain its equation at once from (207) by a proper determination of the parameters that enter. The motion of a ray of light is characterized by the fact that $ds=0$. Let us, then, allow ds to approach zero. In the limit we have, from (170),

$$-\lambda_1\left(\frac{dr}{d\theta}\right)^2 - r^2 + \nu_1\left(\frac{dt}{d\theta}\right)^2 = 0,$$

and from (169)

$$\left(\frac{dt}{d\theta}\right)^2 = \frac{k^2}{h^2}\frac{r^4}{\nu_1^2} \cdot$$

Eliminating $dt/d\theta$ and putting, as before, $r=1/u$, we obtain as the equation of the orbit

$$\left(\frac{du}{d\theta}\right)^2 = -\frac{u^2}{\lambda_1} + \frac{k^2}{h^2}\frac{1}{\lambda_1\nu_1} \cdot$$

The comparison of this equation with (171) shows that the effect of taking $ds=0$ is to cause both h and k to approach infinity in such a way that $k^2/h^2<\infty$. We achieve our purpose, then, by putting $\alpha^2=0$, or, what is equivalent, $\eta=0$. Then (205) becomes

$$U = \beta - \tfrac{1}{2}mb_1\beta^2 + \cdots. \tag{209}$$

This equation can be solved as a power series for β in U, vanishing with U,

$$\beta = U + \tfrac{1}{2}mb_1U^2 + \cdots.$$

The equation of the orbit (207) becomes

$$u = \beta\cos\theta + \tfrac{1}{4}ma_1\beta^2 - \tfrac{1}{2}mb_1\beta^2 - \tfrac{1}{4}ma_1\beta^2\cos 2\theta + \cdots. \tag{210}$$

In the application to the solar field $U^2=1/R^2$ is small, so that powers of β above the second may be neglected. Then (210) can be written in rectangular coördinates as

$$x = \frac{1}{\beta} + \frac{1}{2}mb_1\beta(x^2+y^2)^{1/2} - \frac{1}{2}ma_1\beta y^2(x^2+y^2)^{-1/2}.$$

The equations of the asymptotes to this curve are

$$x = \frac{1}{\beta} \pm \frac{1}{2} m\beta(a_1 - b_1)y,$$

and the angle between them, which we may denote by χ, is

$$\chi = m\beta(a_1 - b_1).$$

To an order of accuracy enormously greater than that of observation we have

$$\beta = U = \frac{1}{R},$$

so that

$$\chi = \frac{m(a_1 - b_1)}{R}. \tag{211}$$

For the Einstein law $a_1 = -b_1 = 2$, which gives

$$\chi = \frac{4m}{R}.$$

This is the well-known result; if R is put equal to the radius of the sun, so that the ray grazes the sun's limb, it is found that $\chi = 1''.75$. For the law (134), or for the law (134) plus (94), we have $a_1 = b_1$, so that $\chi = 0$, a result which we knew in the first place from the form of the solutions of these laws.

The equation (210) gives the path of a light ray for any law of gravitation, i.e. for any set of a_i, b_i which satisfy a gravitational equation under the first three postulates of §34, in case the entire gravitational field in play is attributable to the mass m; in other words, provided that the $g_{\mu\nu}$ reduce to the Galilean values for $m = 0$.

APPENDIX I

We give the proof of the lemma of §30 because of its brevity.

LEMMA. *The k-order determinant D_k whose principal diagonal elements are $1+\rho$, all other elements being 1, has the value*

$$D_k = (\rho + k)\rho^{k-1}.$$

For consider the k-order determinant

$$d_k \equiv \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1+\rho & \cdots & 1 \\ \cdot & \cdot & 1+\rho & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ & \cdot & \cdots & \cdot \\ 1 & 1 & \cdots & 1+\rho \end{vmatrix};$$

to evaluate this we subtract the first row from each other row, whereupon we obtain $d_k = \rho^{k-1}$. If now we expand d_{k+1} by the elements of its first column we have

$$d_{k+1} = D_k - kd_k;$$

hence

$$D_k = d_{k+1} + kd_k = \rho^k + k\rho^{k-1} = (\rho + k)\rho^{k-1}.$$

If $\rho = -m$, we have

$$D_k = (-1)^{k-1}(k - m)m^{k-1}.$$

APPENDIX II

THEOREM I. *If every tensor $A_{(2m-2)}$ obtained from a fundamental non-differential tensor A_{2m} by double contraction of $g^{(2)}A_{(2m)}$ is identically zero then $A_{(2m)}$ is identically zero.*

By the method of §31, with $\rho_{12} = \rho_{21} = 0$ we find

$$A'_{1\cdots 1} = \rho_{11}^{2m} A_{1\cdots 1}.$$

Hence

$$A_{1\cdots 1} = a_{1\cdots 1} g_{11}^{m}.$$

Similarly

$$A_{1\cdots 12} = a_{1\cdots 2} g_{11}^{m-1} g_{12}.$$

Now

$$g^{11} = \frac{g_{22}}{g}, \quad g^{12} = g^{21} = -\frac{g_{12}}{g}, \quad g^{22} = \frac{g_{11}}{g}.$$

By hypothesis

(1) $g^{11}A_{\dots 1 \dots 1 \dots} + g^{12}A_{\dots 1 \dots 2 \dots} + g^{21}A_{\dots 2 \dots 1 \dots} + g^{22}A_{\dots 2 \dots 2 \dots} = 0$

is an identity in $g_{\mu\nu}$.

Let us take $g_{12} = g_{21} = 0$. Then by (1)

$$\frac{1}{g_{11}} A_{\dots 1 \dots 1 \dots} + \frac{1}{g_{22}} A_{\dots 2 \dots 2 \dots} = 0$$

for all $g_{11} \neq 0$, $g_{22} \neq 0$. Hence in particular

$$\frac{1}{g_{11}} A_{1 \dots 1} + \frac{1}{g_{22}} A_{221 \dots 1} = 0\,.$$

But $A_{1 \dots 1} = a_{1 \dots 1} g_{11}{}^m$. Hence $A_{221 \dots 1} = -a_{1 \dots 1} g_{11}^{m-1} g_{22}$.

Let $A_{k/2}$ be any component with k twos as indices. Then similarly

$$A_{2/2} = -\, a_{1 \dots 1}\, g_{11}^{m-1} g_{22}\,.$$

Now again by (1)

$$\frac{1}{g_{11}} A_{\dots 2 \dots 2 \dots 1 \dots 1 \dots} + \frac{1}{g_{22}} A_{\dots 2 \dots 2 \dots 2 \dots 2 \dots} = 0.$$

Thus

$$A_{4/2} = -\frac{g_{22}}{g_{11}} A_{2/2} = +\, a_{1 \dots 1}\, g_{11}^{m-2} g_{22}^{2}\,.$$

Similarly

(2) $$A_{2r/2} = (-1)^r a_{1 \dots 1}\, g_{11}^{m-r} g_{22}^{r} = (-1)^r \alpha\, g_{11}^{m-r} g_{22}^{r}\,.$$

But we must show that by choosing $g_{12} = g_{21} = 0$ we have not changed the value of $A_{2r/2}$:

In general $A'_{2r/2} = \rho_{11}{}^{2m-2r} \rho_{22}^{2r} A_{2r/2}$; hence $A_{2r/2} = c g_{11}^{m-r} g_{22}^{r}$, and this is not changed if we take $g_{12} = g_{21} = 0$. Hence (2) is correct.

Now by (2)

$$g^{11} A_{(0/2)} + g^{22} A_{(2/2) \dots 2 \dots 2 \dots} = 0,$$

so that by (1), with general $g_{\mu\nu}$

(3) $$g^{12} A_{(1/2) \dots 1 \dots 2 \dots} + g^{21} A_{(1/2) \dots 2 \dots 1 \dots} = 0\,;$$

hence

$$A_{(1/2) \dots 2 \dots} = -\, A_{(1/2) \dots 2}$$

Thus

$$A_{(1/2)\cdots 2\cdots 1\cdots 1} = A_{(1/2)\cdots 1\cdots 2\cdots 1};$$

but by (3) these two expressions are of opposite sign; thus $A_{(1/2)}=0$. Now by (1)

$$g^{11}A_{(1/2)} + g^{12}A_{(2/2)} + g^{21}A_{(2/2)} + g^{22}A_{(3/2)} = 0.$$

Thus every

$$A_{(3/2)} = -2\alpha g_{11}{}^{m-2}g_{12}g_{22}.$$

Again by (1)

$$g^{11}A_{(2/2)\cdots 22} + g^{12}A_{(3/2)\cdots 1222} + g^{21}A_{(3/2)\cdots 2122} + g^{22}A_{(4/2)\cdots 2222} = 0,$$

which gives by (2)

$$g^{12}A_{(3/2)\cdots 1222} + g^{21}A_{(3/2)\cdots 2122} = 0;$$

thus $4\alpha g_{11}{}^{m-2}g_{12}^2g_{22}=0$, so that $\alpha=0$, and $A_{2r/2}=A_{1/2}=A_{3/2}=0$; now it follows readily from (1) that $A_{2r+1/2}=0$, so that $A=0$, and the theorem is proved.

THEOREM II. *The number of linearly independent fundamental non-differential tensors of type* $\binom{0}{2m}$ *is*

$$N(m) = 1\cdot 3\ 5\cdot\cdots\cdot(2m-1).$$

A complete set of linearly independent ones is given by m successive multiplications of $g_{(2)}$, *with all possible orders of indices.*

We shall prove this theorem by induction. We have seen in §31 that it is true for $\binom{0}{2}$ and $\binom{0}{4}$; let us assume that it is true for $\binom{0}{2k}$, where $k\leqq m-1$. Then every A_{2m-2} can be expressed as stated.

We now wish to show that

LEMMA I. *For every* A_{2m-2} *there exists exactly one* B_{2m} *which is a sum of terms each of which is a product of m successive multiplications of* $g_{(2)}$, *such that every double contraction of* $g^{(2)}B_{2m}$ *is identically equal to* A_{2m-2}. For example, the general A_4 is, by (76)

$$A_{\rho\mu\nu\sigma} = \alpha g_{\rho\mu}g_{\nu\sigma} + \beta g_{\rho\nu}g_{\mu\sigma} + \gamma g_{\rho\sigma}g_{\mu\nu}, \tag{4}$$

and we wish to find

$$B_{\lambda\tau\rho\mu\nu\sigma} = \sum g_{(2)}g_{(2)}g_{(2)},$$

such that

$$\left\{\begin{aligned} &\sum_{\lambda\tau} g^{\lambda\tau}B_{\lambda\tau\rho\mu\nu\sigma} = A_{\rho\mu\nu\sigma},\\ &\sum_{\lambda\rho} g^{\lambda\rho}B_{\lambda\tau\rho\mu\nu\sigma} = A_{\tau\mu\nu\sigma},\\ &\cdots\cdots\cdots\\ &\sum_{\nu\sigma} g^{\nu\sigma}B_{\lambda\tau\rho\mu\nu\sigma} = A_{\lambda\tau\rho\mu}.\end{aligned}\right. \tag{5}$$

Let

$$(6)\qquad B_{\lambda\tau\rho\mu\nu\sigma} = g_{\lambda\tau}(\alpha_1 g_{\rho\mu}g_{\nu\sigma} + \alpha_2 g_{\rho\nu}g_{\mu\sigma} + \alpha_3 g_{\rho\sigma}g_{\mu\nu}) + g_{\sigma\tau}(\alpha_4 g_{\rho\mu}g_{\nu\lambda} + \alpha_5 g_{\rho\nu}g_{\mu\lambda} + \alpha_6 g_{\rho\lambda}g_{\mu\nu}) + g_{\nu\tau}(\cdots) + g_{\mu\tau}(\cdots) + g_{\rho\tau}(\alpha_{13}g_{\lambda\mu}g_{\nu\sigma} + \alpha_{14}g_{\lambda\nu}g_{\mu\sigma} + \alpha_{15}g_{\lambda\sigma}g_{\mu\nu}).$$

Then (5) becomes

$$(7)\quad f_i(\alpha_j) = \alpha(i = 1, \cdots, 5),\ \beta(i = 6, \cdots, 10),\ \gamma(i = 11, \cdots, 15),$$

where the f_i are linear homogeneous equations with constant coefficients in the 15 quantities α_j. Let (7) have the determinant D. Consider

$$(8)\qquad f_i(\alpha_j) = 0 \quad (i = 1, \cdots, 15).$$

If (8) is satisfied then every first reduction of (6) is identically zero, so that (6) is identically zero by Theorem I above; thus the only solution of (8) is $\alpha_j = 0$, so that $D \neq 0$; hence (7) has a unique solution.

Clearly by induction we can now prove the lemma.

In a similar way we can prove

LEMMA II. *For every set of $k \leqq C_2^{2m}$ tensors $A_{2m-2}^{(k)}$ there exists one tensor B_{2m} which is a sum of terms each of which is a product of m successive multiplications of $g_{(2)}$ such that the tensors obtained by double contraction of $g^{(2)}A_{2m}$ are exactly the tensors $A_{2m-2}^{(k)}$; if the indices are specified then there exists exactly one such B_{2m}.*

We are now in a position to prove Theorem II. For let A_{2m} be a tensor of type $\binom{0}{2m}$. Then there are exactly C_2^{2m} tensors $A_{2m-2}^{(k)}$ obtained by double contraction of $g^{(2)}A_{2m}$. Then by Lemma II there exists exactly one tensor B_{2m} such that for every choice of indices the tensor obtained by double contraction of $g^{(2)}B_{2m}$ is the same as the tensor obtained by double contraction of $g^{(2)}A_{2m}$. Hence, if we set

$$\overline{A}_{2m} \equiv A_{2m} - B_{2m}$$

we see that every tensor $\overline{A}_{2m-2}$ obtained from $\overline{A}_{2m}$ by one reduction with $g^{(2)}$ is identically zero; hence by Theorem I, Appendix I, the tensor $\overline{A}_{2m}$ is identically zero, and hence $A_{2m} = B_{2m}$.

But B_{2m} is of the form required in Theorem 2; hence A_{2m} is of that form also. The value of $N(m)$ is immediately calculated.

APPENDIX III

THEOREM I. *There exists no* $\overline{B}_6$. (Cf. §39.)

By (107) we see that B_6 is the sum of 15 sets of terms of the form

$$(\alpha_{ij}^{ab\cdot cd} g_{ab}g_{cd} + \alpha_{ij}^{ac\cdot bd} g_{ac}g_{bd} + \alpha_{ij}^{ad\cdot bc} g_{ad}g_{bc})E_{ij},$$

where the constant coefficients $\alpha_{ij}^{ab\cdot cd}$ are symmetric in (a, b), in (c, d) and in (i, j) and also

$$\alpha_{ij}^{ab\cdot cd} = \alpha_{ij}^{cd\cdot ab}. \tag{1}$$

If B_6 is to be a $\overline{B}_6$ then by definition every first reduction of B_6 must be identically zero; hence, by §39, the coefficient of each of the six terms of each reduction is zero. If we reduce B_6 by g^{ab} then the coefficient of $g_{cd}E_{ij}$ is

$$4\alpha_{ij}^{ab\cdot cd} + \alpha_{ij}^{ac\cdot bd} + \alpha_{ij}^{ad\cdot bc} + \alpha_{ai}^{bj\cdot cd} + \alpha_{aj}^{bi\cdot cd} + \alpha_{bi}^{aj\cdot cd} + \alpha_{bj}^{ai\cdot cd}.$$

Let us introduce new variables

$$A_{ij}^{ab\cdot cd} = 4\alpha_{ij}^{ab\cdot cd} + \alpha_{ij}^{ac\cdot bd} + \alpha_{ij}^{ad\cdot bc}. \tag{2}$$

Then if B_6 is to be a $\overline{B}_6$ we have

$$A_{ij}^{ab\cdot cd} + \alpha_{ai}^{bj\cdot cd} + \alpha_{aj}^{bi\cdot cd} + \alpha_{bi}^{aj\cdot cd} + \alpha_{bj}^{ai\cdot cd} = 0. \tag{3}$$

From (3) we have

$$A_{ab}^{ij\cdot cd} = A_{ij}^{ab\cdot cd};$$

and by (2) and (1) we have

$$A_{ij}^{ab\cdot cd} = A_{ij}^{cd\cdot ab};$$

thus

$$A_{ij}^{ab\cdot cd} = A_{ij}^{cd\cdot ab} = A_{ab}^{ij\cdot cd} = A_{ab}^{cd\cdot ij} = A_{cd}^{ab\cdot ij} = A_{cd}^{ij\cdot ab}, \tag{4}$$

so that there are only 15 distinct A's. The solution of (2) is

$$18\alpha_{ij}^{ab\cdot cd} = 5A_{ij}^{ab\cdot cd} - A_{ij}^{ac\cdot bd} - A_{ij}^{ad\cdot bc}; \tag{5}$$

substituting this in (3), and using (4), we have

$$18A_{ij}^{ab\cdot cd} + 10(A_{ai}^{bj\cdot cd} + A_{bi}^{aj\cdot cd}) - (A_{ai}^{bc\cdot dj} + A_{ai}^{bd\cdot cj} + A_{bi}^{ac\cdot dj} + A_{bi}^{ad\cdot cj}$$
$$+ A_{ci}^{ad\cdot bj} + A_{ci}^{aj\cdot bd} + A_{di}^{ac\cdot bj} + A_{di}^{aj\cdot bc}) = 0.$$

If we interchange a with c and b with d and add, we obtain

$$(6)\quad 18A_{ij}^{ab\cdot cd} + 5(A_{ai}^{bj\cdot cd} + A_{bi}^{aj\cdot cd} + A_{ci}^{ab\cdot dj} + A_{di}^{ab\cdot cj})$$
$$- (A_{ai}^{bc\cdot dj} + A_{ai}^{bd\cdot cj} + A_{bi}^{ac\cdot dj} + A_{bi}^{ad\cdot cj} + A_{ci}^{ad\cdot bj} + A_{ci}^{aj\cdot bd}$$
$$+ A_{di}^{ac\cdot bj} + A_{di}^{aj\cdot bc}) = 0.$$

These are 15 homogeneous equations in the 15 quantities A. Let us number these equations (ij, ab, cd) by the first term; let us now number the rows and columns of the symmetric determinant D of (6) as follows: the row (ij, ab, cd) corresponds to the equation of the same number; the column (ij, ab, cd) corresponds to the terms in $A_{ij}^{ab\,cd}$. We can now readily see that the elements of D are as follows: Let the element in the row number r and column number c be D_{rc}; then

$$\text{when } r, c \text{ have } \begin{cases} \text{3 pair alike} & D_{rc}=18 \quad \text{(diagonal)}, \\ \text{only lower pair alike} & D_{rc}=0 \quad \text{(2 elements)}, \\ \text{only one upper pair alike} & D_{rc}=5 \quad \text{(4 elements)}, \\ \text{no pair alike} & D_{rc}=-1 \quad \text{(8 elements)}; \end{cases}$$

this determinant we shall call $D(18, 0, 5, -1)$; we wish to show that it is not zero. To facilitate† this we introduce the

LEMMA. *If*

$$(7)\qquad \sum_{i=1}^{n} a_{ij} = k \neq 0 \qquad (j = 1, \cdots, n)$$

then $\det a_{ij} = 0$ *implies that* $\det (a_{ij} - c) = 0$ *for every constant* c.

For consider the equations

$$(8)\qquad \sum_{j=1}^{n} a_{ij}x_j = 0 \qquad (i = 1, \cdots, n);$$

then

$$\sum_{i,j=1}^{n} a_{ij}x_j = \sum_{j=1}^{n} x_j \sum_{i=1}^{n} a_{ij} = k\sum_{j=1}^{n} x_j = 0,$$

so that

$$\sum_{j=1}^{n} x_j = 0;$$

hence

$$\sum_{j=1}^{n} cx_j = 0;$$

subtracting this from (8) we have

$$(9)\qquad \sum_{j=1}^{n} (a_{ij} - c)x_j = 0 \qquad (i = 1, \cdots, n).$$

† Although in this case the introduction of this lemma makes no essential change in the proof that $D\neq 0$, we have introduced it because there are many cases in which it is very useful in similar proofs.

Now every solution of (8) is also a solution of (9); if det $a_{ij}=0$ then (8) has a solution with the x_j not all zero and hence (9) has such a solution; thus det $(a_{ij}-c)=0$. ∎

Our determinant $D(18, 0, 5, -1)$ is one to which this lemma applies, with $k=30$; hence if $D(18, 0, 5, -1)$ is zero then so is $D(19, 1, 6, 0)$. It is easily verified, since D is symmetric, that

$$(10) \qquad D(\alpha,\beta,\gamma,0)\cdot D(19,1,6,0) = D(\bar\alpha,\bar\beta,\bar\gamma,\bar\delta),$$

where

$$(11) \qquad \begin{cases} \bar\alpha = 19\alpha + 2\beta + 24\gamma, \\ \bar\beta = \alpha + 20\beta , \\ \bar\gamma = 6\alpha + 25\gamma, \\ \bar\delta = 6\beta + 7\gamma. \end{cases}$$

If we choose

$$\alpha = -\frac{7}{33}, \quad \beta = \frac{2}{33}, \quad \gamma = \frac{1}{11},$$

then

$$\bar\alpha = -\frac{19}{11}, \quad \bar\beta = \bar\gamma = \bar\delta = 1.$$

Now $D(\bar\alpha, \bar\beta, \bar\gamma, \bar\delta)$ has its principal diagonal elements $-19/11$, and all its other elements equal to 1; hence by the lemma of Appendix I it is not zero; hence by (10) we have $D(19, 1, 6, 0)\neq 0$, so that $D(18, 0, 5, -1)\neq 0$. Now it follows from (6) and (4) that all the $A_{ij}^{ab\,cd}$ are zero, and hence by (5) all the $\alpha_{ij}^{ab\,cd}$ are zero. Thus B_6 is identically zero and there exists no $\overline{B}_6$.

THEOREM II. *There exists no $\overline{B}_8$.*

The proof of this theorem is similar to the above and will not be given here.

PROPOSITION II. *There exists no $\overline{B}_{2m}$ for $m>4$.* (Cf. §39.)

From (107) we see that

$$B_{2m} = \sum g_{(2)} \overset{\underline{m-1}}{\cdots} g_{(2)}E_{(2)}.$$

By Theorem 10 of §31 we see that this expression contains $C_2^{2m}N(m-1) = mN(m)$ constant coefficients. There are C_2^{2m} first reductions of B_{2m}, each of which contains $(m-1)N(m-1)$ coefficients which are linear homogeneous functions with constant coefficients of the coefficients of B_{2m}. If B_{2m} is to be a $\overline{B}_{2m}$ then all its first reductions must be identically zero; if we use induction, assuming that the proposition is correct for $m-1$, then every coefficient of every first reduction of B_{2m} is zero. We are thus led to a set of linear homogeneous equations with constant coefficients in the $mN(m)$ constants of B_{2m}, the number of such equations being

$C_2^{2m}(m-1)N(m-1)$. Thus there are $(m-1)$ times as many equations as there are variables (the constants of B_{2m}); the conditions which these variables must satisfy become stronger as m increases. We have seen that for $m=3, 4$ the conditions are so strong that all the variables must be zero; it seems very likely, therefore, that with m still greater the same will be true, so that there exists no $\overline{B}_{2m}$ for $m>4$. We have not been able to construct a rigorous proof of this.

APPENDIX IV

THEOREM. *There exists no* $\overline{A}_8$. (Cf. §40.)

By (107) we see that A_8 is a sum of terms of the form

$$(1)\quad (\alpha_{1342}g_{\mu_5\mu_6}g_{\mu_7\mu_8}+\beta_{1342}g_{\mu_5\mu_7}g_{\mu_6\mu_8}+\gamma_{1342}g_{\mu_5\mu_8}g_{\mu_6\mu_7})(W_{\mu_1\mu_3\mu_4\mu_2}+W_{\mu_1\mu_4\mu_3\mu_2})\ ;$$

the number of such terms in A_8 with distinct indices is $C_4^8=70$; hence there are 210 constants α, β, γ. If this is to be an $\overline{A}_8$ then by definition every one of its $C_2^8=28$ first reductions must be identically zero. Each first reduction is the sum of 15 terms as in (117) and 15 more with W replaced by $\breve{W}$; for this sum to be identically zero each of these 30 terms must be zero; thus each first reduction of A_8 leads to 30 linear homogeneous equations in the α, β, γ. Hence we obtain a set of 840 linear homogeneous equations in the 210 quantities α, β, γ. To obtain a set of equations whose solubility is feasible we take A_8 equal to a sum of terms in which, as in the case of A_6 of §40, the indices are arranged in a definite way. We shall represent the term (1) by the symbol 5786−1342; in general, the symbol $abcd-ijkl$ shall represent the term

$$(\alpha_{ijkl}g_{\mu_a\mu_d}g_{\mu_b\mu_c}+\beta_{ijkl}g_{\mu_a\mu_b}g_{\mu_d\mu_c}+\gamma_{ijkl}g_{\mu_a\mu_c}g_{\mu_d\mu_b})(W_{\mu_i\mu_j\mu_k\mu_l}+W_{\mu_i\mu_k\mu_j\mu_l}).$$

We now take A_8 equal to the sum of the 70 terms

	*5786−1342	5786−1342	5786−1342
	3784−5126	3486−5127	3746−5128
	1782−3564	1286−3574	1726−3584
	*3786−1254	3786−1254	3786−1254
	2785−3146	2586−3147	2756−3148
	1784−2365	1486−2375	1746−2385
	4785−1263	4586−1273	4756−1283
	*2786−4135	2786−4135	2786−4135
(2)	1783−2456	1386−2457	1736−2458
	3785−1246	3586−1247	3756−1248
	2784−3165	2486−3175	2746−3185
	*1786−2354	1786−2354	1786−2354
	*4786−1235	4786−1235	4786−1235
	2783−4156	2386−4157	2736−4158
	1785−2463	1586−2473	1756−2483.

By (2) we mean the sum of its 35 distinct terms (noting that each of the starred lines contains two repetitions), plus the sum of the 35 terms obtained from these by transposing the right and left sets of indices. Thus, for example, the first line of the first column represents the term written plus the term $1342-5786$. When we wish to consider these terms we shall always refer to them as *transposed.* The array (2) is an extension of (117) of §40, and is obtained from it as follows: to get the first column we place 78 between the two digits on the left of (117); to get the second column we place 87 after the first two digits on the left of (117), and then interchange 6 and 7; to get the third column we place 78 in the second and fourth places on the left of (117), and then interchange 6 and 8.

When we reduce (2) with $g^{\lambda\tau}$ the indices λ, τ are called the *reducing indices*; the term $abcd-ijkl$ is called the *base* when λ, τ are chosen from a, b, c, d. For each reduction we specify the reducing indices and the base, and then consider only the coefficients of W_{ijkl} and W_{ikjl}, using the relations (50), as we did in §40; when we equate this reduction identically to zero we obtain the two equations stating that these two coefficients are zero; thus each base yields two equations for each reduction. Each of these equations contains three terms derived from the base and, as in §40, eight others.

We encounter in the reductions three types of terms, as defined in §40. In the previous case $(m=3)$ it was possible to construct (117) such that terms of type 3 enter only from the base. In this case such a construction is not possible; however, sets 1 to 4 below of reduction equations have this property. Furthermore, the properties of (2) are such that we can solve the 840 reduction equations in several successive sets, the largest numerical determinant to be evaluated being of order 15. For brevity we shall indicate these sets in tabular form, giving the choice of bases and reducing indices that give each set of equations.

SET 1	*Base*	*Reducing indices of the base*
	Each term of column 1	1st and 4th digits

For example, the first base is $5786-1342$ and the reducing indices are 5, 6. There are thus 15 terms used as base, leading to 30 equations. This set is clearly the same as that obtained from (117) except that here each equation has the factor g_{78} (which can be omitted since it is not identically zero) and that the ϕ_i are different, namely $\phi_i=4\alpha_i+\beta_i+\gamma_i$, so that with the notation of §40, Set 1 is

$$
\begin{aligned}
&4\alpha_i + \beta_i + \gamma_i + f_i(\alpha_j) = 0 \\
(3)\qquad &4\alpha_i + \beta_i + \gamma_i + \check{f}_i(\alpha_j) = 0,
\end{aligned}
$$

from which

$$f_i(\alpha_j) - \check{f}_i(\alpha_j) = 0, \tag{4}$$

whose solution is given in Lemma I of §40.

SET 2	*Base*	*Reducing indices*
	Each term of column 2	1st and 2nd digits

We obtain the same set of equations, but with a cyclic interchange of $\alpha_i, \beta_i, \gamma_i$, so that

$$\begin{aligned} \alpha_i + 4\beta_i + \gamma_i + f_i(\beta_j) &= 0 \\ \alpha_i + 4\beta_i + \gamma_i + \hat{f}_i(\beta_j) &= 0, \end{aligned} \tag{5}$$

$$f_i(\beta_j) - \check{f}_i(\beta_j) = 0. \tag{6}$$

SET 3	*Base*	*Reducing indices*
	Each term of column 3	1st and 3rd digits

In this case we get

$$\begin{aligned} \alpha_i + \beta_i + 4\gamma_i + f_i(\gamma_j) &= 0 \\ \alpha_i + \beta_i + 4\gamma_i + \check{f}_i(\gamma_j) &= 0, \end{aligned} \tag{7}$$

$$f_i(\gamma_j) - \check{f}_i(\gamma_j) = 0. \tag{8}$$

The solutions of equations (4), (6) and (8) are given in Lemma I of §40. It is to be understood that in this case, as in §40, we have numbered the variables α, β, γ by lines of the table; but here the α_i, for example, represent different variables in the sets 1 to 3. For example in Set 1, $\alpha_2 \equiv \alpha_{5126}$, while in Set 2, $\alpha_2 \equiv \alpha_{5127}$, etc. From now on we shall drop this condensed notation in the interest of clarity.

By Lemma I of §40 the solution of (4) is

$$\begin{aligned} \alpha_{5126} &= -\alpha_{3564} + \alpha_{2456} + \alpha_{3165} \\ \alpha_{3146} &= \alpha_{1342} - \alpha_{1254} + \alpha_{4156} \\ \alpha_{2365} &= \alpha_{2456} + \alpha_{3165} - \alpha_{4156} \\ \alpha_{1263} &= \alpha_{1342} - \alpha_{3564} - \alpha_{1254} + \alpha_{2456} + \alpha_{3165} \\ \alpha_{4135} &= \alpha_{3564} + \alpha_{1254} - \alpha_{2456} \\ \alpha_{1246} &= \alpha_{1342} - \alpha_{3564} - \alpha_{1254} + \alpha_{2456} + \alpha_{4156} \\ \alpha_{2354} &= \alpha_{3564} + \alpha_{1254} - \alpha_{4156} \\ \alpha_{1235} &= \alpha_{1254} + \alpha_{3165} - \alpha_{4156} \\ \alpha_{2463} &= \alpha_{1342} - \alpha_{1254} + \alpha_{2456}. \end{aligned} \tag{9}$$

The solutions of (6) and (8) are obtained from (9) by a cyclic interchange of the variables α, β, γ and by a simultaneous cyclic interchange of the indices 6, 7, 8. If we substitute these three sets of equations into (3), we obtain

$$
\begin{aligned}
&{}^{*}\gamma_{1342} &= &- \beta_{1342} - 2\alpha_{1342} + \alpha_{2456} + \alpha_{3165}\\
&\gamma_{5126} &= &- \beta_{5126} + \alpha_{1342} + 3\alpha_{3564} - 2\alpha_{2456} - 2\alpha_{3165}\\
&\gamma_{3564} &= &- \beta_{3564} + \alpha_{1342} - 3\alpha_{3564} + \alpha_{2456} + \alpha_{3165}\\
&{}^{*}\gamma_{1254} &= &- \beta_{1254} + \alpha_{1342} - 3\alpha_{1254} + \alpha_{2456} + \alpha_{3165}\\
&\gamma_{3146} &= &- \beta_{3146} - 2\alpha_{1342} + 3\alpha_{1254} + \alpha_{2456} + \alpha_{3165} - 3\alpha_{4156}\\
&\gamma_{2365} &= &- \beta_{2365} + \alpha_{1342} - 2\alpha_{2456} - 2\alpha_{3165} + 3\alpha_{4156}\\
&\gamma_{1263} &= &- \beta_{1263} - 2\alpha_{1342} + 3\alpha_{3564} + 3\alpha_{1254} - 2\alpha_{2456} - 2\alpha_{3165}\\
(10)\ &{}^{*}\gamma_{4135} &= &- \beta_{4135} + \alpha_{1342} - 3\alpha_{3564} - 3\alpha_{1254} + 4\alpha_{2456} + \alpha_{3165}\\
&\gamma_{2456} &= &- \beta_{2456} + \alpha_{1342} - 2\alpha_{2456} + \alpha_{3165}\\
&\gamma_{1246} &= &- \beta_{1246} - 2\alpha_{1342} + 3\alpha_{3564} + 3\alpha_{1254} - 2\alpha_{2456} + \alpha_{3165} - 3\alpha_{4156}\\
&\gamma_{3165} &= &- \beta_{3165} + \alpha_{1342} - \alpha_{5126} - \alpha_{3564} + 2\alpha_{2456} - \alpha_{3165}\\
&{}^{*}\gamma_{2354} &= &- \beta_{2354} + \alpha_{1342} - 3\alpha_{3564} - 3\alpha_{1254} + \alpha_{2456} + \alpha_{3165} + 3\alpha_{4156}\\
&{}^{*}\gamma_{1235} &= &- \beta_{1235} + \alpha_{1342} - 3\alpha_{1254} + \alpha_{2456} - 2\alpha_{3165} + 3\alpha_{4156}\\
&\gamma_{4156} &= &- \beta_{4156} + \alpha_{1342} - \alpha_{5126} - \alpha_{3564} + 2\alpha_{2456} + 2\alpha_{3165} - 3\alpha_{4156}\\
&\gamma_{2463} &= &- \beta_{2463} - 2\alpha_{1342} + 3\alpha_{1254} - 2\alpha_{2456} + \alpha_{3165}.
\end{aligned}
$$

The corresponding sets obtained from (5) and (7) are obtained from the set (10), as in the case of (9), by cyclic interchanges of the variables α, β, γ, and of the indices 6, 7, 8. The asterisks indicate those equations in which the variable on the left occurs in a starred line of (2).

The 18 variables given on the right sides of (9) and its two related sets are entirely arbitrary up to this point. We now show that there are fifteen relations between these variables that effectively reduce the arbitrary variables to three. These relations are the starred equations of (10) and its two related sets, and may be written, after further elimination by (9):

$$
\begin{aligned}
&\gamma_{1342} + \beta_{1342} + 2\alpha_{1342} - \alpha_{2456} - \alpha_{3165} = 0\\
&\alpha_{1342} + \gamma_{1342} + 2\beta_{1342} - \beta_{2457} - \alpha_{3175} = 0\\
&\beta_{1342} + \alpha_{1342} + 2\gamma_{1342} - \gamma_{2458} - \gamma_{3185} = 0\\
&\gamma_{1254} + \beta_{1254} + \alpha_{1342} + 3\alpha_{1254} - \alpha_{2456} - \alpha_{3165} = 0\\
&\alpha_{1254} + \gamma_{1254} + \beta_{1342} + 3\beta_{1254} - \beta_{2457} - \beta_{3175} = 0\\
&\beta_{1254} + \alpha_{1254} + \gamma_{1342} + 3\gamma_{1254} - \gamma_{2458} - \gamma_{3185} = 0\\
&\gamma_{3584} + \gamma_{1254} - \gamma_{2458} + \beta_{3574} + \beta_{1254} - \beta_{2457} - \alpha_{1342} + 3\alpha_{3564} + 3\alpha_{1254}\\
&\qquad - 4\alpha_{2456} - \alpha_{3165} = 0\\
&\gamma_{3584} + \gamma_{1254} - \gamma_{2458} + \alpha_{3564} + \alpha_{1254} - \alpha_{2456} - \beta_{1342} + 3\beta_{3574} + 3\beta_{1254}\\
&\qquad - 4\beta_{2457} - \beta_{3175} = 0
\end{aligned}
$$

$$\beta_{3574} + \beta_{1254} - \beta_{2457} + \alpha_{3564} + \alpha_{1254} - \alpha_{2456} - \gamma_{1342} + 3\gamma_{3584} + 3\gamma_{1254} - 4\gamma_{2458} - \gamma_{3185} = 0$$

$$\gamma_{3584} + \gamma_{1254} - \gamma_{4158} + \beta_{3574} + \beta_{1254} - \beta_{4157} - \alpha_{1342} + 3\alpha_{3564} + 3\alpha_{1254} - \alpha_{2456} - \alpha_{3165} - 3\alpha_{4156} = 0$$

$$\gamma_{3584} + \gamma_{1254} - \gamma_{4158} + \alpha_{3564} + \alpha_{1254} - \alpha_{4156} - \beta_{1342} + 3\beta_{3574} + 3\beta_{1254} - \beta_{2457} - \beta_{3175} - 3\beta_{4157} = 0$$

$$\beta_{3574} + \beta_{1254} - \beta_{4157} + \alpha_{3564} + \alpha_{1254} - \alpha_{4156} - \gamma_{1342} + 3\gamma_{3584} + 3\gamma_{1254} - \gamma_{2458} - \gamma_{3185} - 3\gamma_{4158} = 0$$

$$\gamma_{1254} + \gamma_{3185} - \gamma_{4158} + \beta_{1254} + \beta_{3175} - \beta_{4157} - \alpha_{1342} + 3\alpha_{1254} - \alpha_{2456} + 2\alpha_{3165} - 3\alpha_{4156} = 0$$

$$\gamma_{1254} + \gamma_{3185} - \gamma_{4158} + \alpha_{1254} + \alpha_{3165} - \alpha_{4156} - \beta_{1342} + 3\beta_{1254} - \beta_{2457} + 2\beta_{3175} - 3\beta_{4157} = 0$$

$$\beta_{1254} + \beta_{3175} - \beta_{4157} + \alpha_{1254} + \alpha_{3165} - \alpha_{4156} - \gamma_{1342} + 3\gamma_{1254} - \gamma_{2458} + 2\gamma_{3185} - 3\gamma_{4158} = 0.$$

The matrix of this set of equations is of rank 15, and the solution is

$$\begin{aligned}
\alpha_{3165} &= \alpha_{4156} = \alpha_{2456} = \alpha_{3564} \equiv \alpha \\
\beta_{3175} &= \beta_{4157} = \beta_{2457} = \beta_{3574} \equiv \beta \\
\gamma_{3185} &= \gamma_{4158} = \gamma_{2458} = \gamma_{3584} \equiv \gamma \\
\alpha_{1342} &= \alpha_{1254} = \tfrac{3}{2}\alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma \equiv \delta_1 \\
\beta_{1342} &= \beta_{1254} = -\tfrac{1}{2}\alpha + \tfrac{3}{2}\beta - \tfrac{1}{2}\gamma \equiv \delta_2 \\
\gamma_{1342} &= \gamma_{1254} = -\tfrac{1}{2}\alpha - \tfrac{1}{2}\beta + \tfrac{3}{2}\gamma \equiv \delta_3,
\end{aligned}$$

where α, β, γ, δ_1, δ_2, δ_3 are defined as here indicated.

We substitute this solution in (10) and its related sets. Before doing this, however, we introduce one other notation that is almost required in the sequel. We shall denote by $\alpha_{(6)}$ the set of α_{ijkl} that contain exactly one subscript six, and neither of the subscripts 7 and 8, by $\gamma_{(7)}$ the set of γ_{ijkl} with exactly one subscript seven, and neither of the subscripts 6 and 8, etc. The same holds for $\alpha_{(67)}$ or $\beta_{(678)}$, for example. An asterisk attached to a variable will indicate that no 6, 7, or 8 is among the subscripts. We may then write the result of this substitution as

$$\begin{aligned}
&\alpha^* = \delta_1; \quad \beta^* = \delta_2; \quad \gamma^* = \delta_3, \\
&\alpha_{(6)} = \alpha; \quad \beta_{(6)} = \beta; \quad \gamma_{(6)} = \gamma, \\
(11) \qquad &\gamma_{(6)} = -\beta_{(6)} + \eta_1, \\
&\alpha_{(7)} = -\gamma_{(7)} + \eta_2, \\
&\beta_{(8)} = -\alpha_{(8)} + \eta_3,
\end{aligned}$$

where

$$\eta_1 \equiv \tfrac{1}{2}\alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma,$$
$$\eta_2 \equiv -\tfrac{1}{2}\alpha + \tfrac{1}{2}\beta - \tfrac{1}{2}\gamma,$$
$$\eta_3 \equiv -\tfrac{1}{2}\alpha - \tfrac{1}{2}\beta + \tfrac{1}{2}\gamma.$$

SET 4	*Base*	*Reducing indices*
	(a) Each starred term of column 1	6, 7
	(b) Each starred term of column 1	6, 8
	(c) Each starred term of column 1	7, 8

We have 5 bases each with 3 reductions; hence we obtain 30 equations. In writing these equations we shall express them by (11) in terms of only δ_i, η_i, $\beta_{(6)}$, $\alpha_{(7)}$, $\alpha_{(8)}$. The first set (a) may be written (noting that each term on the left is of type 1 or 2 (§40)):

(12a)
$$
\begin{aligned}
&-2\beta_{3146} + \beta_{1263} + \beta_{1246} - 2\beta_{2463} - 2\alpha_{3147} + \alpha_{1273} + \alpha_{1247} - 2\alpha_{2473} \\
&\qquad = -\delta_1 - \delta_2 - 4\delta_3 \\
&\beta_{3146} - 2\beta_{1263} - 2\beta_{1246} + \beta_{2463} + \alpha_{3147} - 2\alpha_{1275} - 2\alpha_{1247} + \alpha_{2473} \\
&\qquad = -\delta_1 - \delta_2 - 4\delta_3 \\
&2\beta_{1246} - \beta_{5126} - 2\beta_{4156} + \beta_{2456} + 2\alpha_{1247} - \alpha_{5127} - \alpha_{4157} + \alpha_{2457} \\
&\qquad = -\delta_1 - \delta_2 - 4\delta_3 + \eta_1 + \eta_2 \\
&\text{(conjugate of above)} = -\delta_1 - \delta_2 - 4\delta_3 + \eta_1 + \eta_2 \\
&-\beta_{3146} - \beta_{3165} - 2\beta_{3564} + 2\beta_{4156} - \alpha_{3147} - \alpha_{3175} - 2\alpha_{3574} + 2\alpha_{4157} \\
&\qquad = -\delta_1 - \delta_2 - 4\delta_3 \\
&\text{(conjugate of above)} = -\delta_1 - \delta_2 - 4\delta_3 + 3\eta_1 + 3\eta_2 \\
&2\beta_{2365} - \beta_{2463} - \beta_{2456} + 2\beta_{3564} + 2\alpha_{2375} - \alpha_{2473} - \alpha_{2457} + 2\alpha_{3574} \\
&\qquad = -\delta_1 - \delta_2 - 4\delta_3 + 2\eta_1 + 2\eta_2 \\
&\text{(conjugate of above)} = -\delta_1 - \delta_2 - 4\delta_3 + 2\eta_1 + 2\eta_2 \\
&2\beta_{1263} + \beta_{5126} + \beta_{3165} - 2\beta_{2365} + 2\alpha_{1273} + \alpha_{5127} + \alpha_{3175} - 2\alpha_{2375} \\
&\qquad = -\delta_1 - \delta_2 - 4\delta_3 + 2\eta_1 + 2\eta_2 \\
&\text{(conjugate of above)} = -\delta_1 - \delta_2 - 4\delta_3 - \eta_1 - \eta_2
\end{aligned}
$$

The second set (b) may be written:

$$2\beta_{3146} - \beta_{1263} - \beta_{1246} + 2\beta_{2463} - 2\alpha_{3148} + \alpha_{1283} + \alpha_{1248} - 2\alpha_{2483}$$
$$= -\delta_1 - 4\delta_2 - \delta_3 + 2\eta_1$$
$$(\text{conjugate of above}) = -\delta_1 - 4\delta_2 - \delta_3 + 2\eta_1$$
$$-2\beta_{1246} + \beta_{5126} + 2\beta_{4156} - \beta_{2456} + 2\alpha_{1248} - \alpha_{5128} - 2\alpha_{4158} + \alpha_{2458}$$
$$= -\delta_1 - 4\delta_2 - \delta_3 + \eta_1 + \eta_3$$
$$(\text{conjugate of above}) = -\delta_1 - 4\delta_2 - \delta_3 + \eta_1 + \eta_3$$
(12b) $$\beta_{3146} + \beta_{3165} + 2\beta_{3564} - 2\beta_{4156} - \alpha_{3148} - \alpha_{3185} - 2\alpha_{3584} + 2\alpha_{4158}$$
$$= -\delta_1 - 4\delta_2 - \delta_3 + 2\eta_1$$
$$(\text{conjugate of above}) = -\delta_1 - 4\delta_2 - \delta_3 - \eta_1 + 3\eta_3$$
$$-2\beta_{2365} + \beta_{2463} + \beta_{2456} - 2\beta_{3564} + 2\alpha_{2385} - \alpha_{2483} - \alpha_{2458} + 2\alpha_{3584}$$
$$= -\delta_1 - 4\delta_2 - \delta_3 + 2\eta_3$$
$$(\text{conjugate of above}) = -\delta_1 - 4\delta_2 - \delta_3 + 2\eta_3$$
$$-2\beta_{1263} - \beta_{5126} - \beta_{3165} + 2\beta_{2365} + 2\alpha_{1283} + \alpha_{5128} + \alpha_{3185} - 2\alpha_{2385}$$
$$= -\delta_1 - 4\delta_2 - \delta_3 + 2\eta_3$$
$$(\text{conjugate of above}) = -\delta_1 - 4\delta_2 - \delta_3 + 3\eta_1 - \eta_3$$

The third set (c) will not be written explicitly, as it follows from the other two by simple considerations of symmetry.

The sum of the equations (12a) is

$$\delta_1 + \delta_2 + 4\delta_3 + \eta_1 + \eta_2 = 0,$$

or

$$\alpha + \beta - 6\gamma = 0.$$

Similarly the sum of equations (12b) is

$$\alpha - 6\beta + \gamma = 0.$$

Equations (12c), which are not written, give

$$-6\alpha + \beta + \gamma = 0.$$

The determinant of these three equations is distinct from zero, and they admit the unique solution $\alpha = \beta = \gamma = 0$. Then our results (11) become

$$\alpha^* = \beta^* = \gamma^* = \alpha_{(6)} = \beta_{(7)} = \gamma_{(8)} = 0,$$

(13) $$\gamma_{(6)} = -\beta_{(6)},\quad \gamma_{(7)} = -\alpha_{(7)},\quad \beta_{(8)} = -\alpha_{(8)}.$$

From now on we shall use these results to eliminate the $\gamma_{(6)}$, $\gamma_{(7)}$, and $\beta_{(8)}$ from all equations.

We have shown that 45 of the 210 unknowns are zero, and have expressed 30 of those remaining in terms of another 30.

We consider equations (12a), (12b), and (12c), which have now become homogeneous in $\beta_{(6)}$, $\alpha_{(7)}$, $\alpha_{(8)}$. The set (12c) is now a consequence of the other two. We shall solve (12a) for the $\alpha_{(7)}$ in terms of the $\beta_{(6)}$, and (12b) for the $\alpha_{(8)}$ in terms of the $\beta_{(6)}$. Now, due to (13), the set (12b) is obtained from (12a) by replacing the index 7 by 8, and changing the sign of the term in each case. Furthermore, we shall show later that $\alpha_{(7)} = -\alpha_{(8)}$, so that it will be necessary to consider only (12a). The equations may be written

$$\begin{aligned}
\alpha_{1273} + \alpha_{1247} &= -\beta_{1263} - \beta_{1246} \\
\alpha_{4157} - \alpha_{1247} &= -\beta_{4156} + \beta_{1246} \\
\alpha_{3574} - \alpha_{4157} &= -\beta_{3564} + \beta_{4156} \\
-\alpha_{3574} - \alpha_{2375} &= \beta_{3564} + \beta_{2365} \\
-\alpha_{1273} + \alpha_{2375} &= \beta_{1263} - \beta_{2365} \\
\alpha_{3147} + \alpha_{2473} &= -\beta_{3146} - \beta_{2463} \\
-\alpha_{3147} - \alpha_{3175} &= \beta_{3146} + \beta_{3165} \\
\alpha_{5127} + \alpha_{3175} &= -\beta_{5126} - \beta_{3165} \\
-\alpha_{5127} + \alpha_{2457} &= \beta_{5126} - \beta_{2456} \\
-\alpha_{2473} - \alpha_{2457} &= \beta_{2463} + \beta_{2456} .
\end{aligned}$$

The first five equations are entirely independent of the last five. The matrix of coefficients of the left sides is of rank four, and the equations are consistent. The same is true of the last five. The solutions are

$$(14)\quad \begin{aligned}
\alpha_{1247} &= a_2 & \alpha_{3147} &= a_3 \\
\alpha_{1273} &= -a_2 - \beta_{1246} - \beta_{1263} & \alpha_{2473} &= -a_3 - \beta_{3146} - \beta_{2463} \\
\alpha_{4157} &= a_2 + \beta_{1246} - \beta_{4156} & \alpha_{3175} &= -a_3 - \beta_{3146} - \beta_{3165} \\
\alpha_{3574} &= a_2 + \beta_{1246} - \beta_{3564} & \alpha_{5127} &= a_3 + \beta_{3146} - \beta_{5126} \\
\alpha_{2375} &= -a_2 - \beta_{1246} - \beta_{2365} & \alpha_{2457} &= a_3 + \beta_{3146} - \beta_{2456},
\end{aligned}$$

where a_2 and a_3 are arbitrary constants.

We shall now show that $\alpha_{(7)} = -\alpha_{(8)}$, as stated above. To this end consider.

SET 5	*Base*	*Reducing indices*
	Each non-starred term of column 1 transposed.	All pairs not containing 6.

For example, base 5126−3784, reducing indices 5, 1 and 5, 2 and 1, 2. There are 10 bases, each with 3 reductions, so that we get 60 equations. If in this set we subtract each equation from its conjugate we obtain ten

equations $\alpha_{(7)} = -\alpha_{(8)}$, ten equations $\gamma_{(7)} = -\beta_{(8)}$, (which was implied by $\alpha_{(7)} = -\alpha_{(8)}$), and ten other equations which are in fact a consequence of the first twenty.

With the results that we now have it is an easy matter to show that there exists a unique solution of the 840 reduction equations, in which each of the 210 unknowns is zero. We shall be able to sketch the remaining steps only sufficiently to indicate the procedure.

Set 6	*Base*	*Reducing indices*
	Each non-starred term of column 1.	All pairs containing 7 but not 8.

For example, base 3784−5126, reducing indices 3, 7 and 7, 4. There are 40 equations. We may write these equations with the left members containing only $\alpha_{(67)}$, $\beta_{(67)}$, $\gamma_{(67)}$, and the right members only $\alpha_{(7)}$ and $\beta_{(6)}$. They divide into two equal sets, the forms of the left members of which are indicated by the examples

$$\beta_{1726} = \alpha_{3175} + \alpha_{2375} + \alpha_{1273}$$

$$\alpha_{1756} + \beta_{2756} = \alpha_{3175} + \alpha_{2375} - \beta_{5126} - 3\alpha_{5127}.$$

The twenty equations of the second type contain 5 triples, each determining 3 of the (67) variables in terms of $\alpha_{(7)}$, $\beta_{(6)}$; the five remaining equations may be combined with ten of the first set to eliminate the (67) variables, yielding 5 equations between the $\alpha_{(7)}$ and the $\beta_{(6)}$. Now the remaining 10 equations of the first set contain only 5 of the (67) variables, and thus yield an additional 5 equations between the $\alpha_{(7)}$ and the $\beta_{(6)}$. We note also in passing that the equations of Set 6 determine all of the (67) variables homogeneously in terms of $\alpha_{(7)}$ and $\beta_{(6)}$, so that if we show that all of these latter are zero, then all of the former are likewise zero.

We have, then, ten equations between the $\beta_{(6)}$ and the $\alpha_{(7)}$. We eliminate the latter variables by (14), and solve the resulting set for the $\beta_{(6)}$ in terms of a_2 and a_3, obtaining

$$\begin{aligned}
&\beta_{5126} = -\frac{3}{10}(a_2 + a_3) && \beta_{2456} = a_3\\
&\beta_{3564} = -(a_2 - a_3) && \beta_{1246} = -\frac{3}{5}(a_2 + a_3)\\
(15)\qquad &\beta_{3146} = -\frac{1}{5}(a_2 + a_3) && \beta_{3165} = -\frac{1}{5}(a_2 - 4a_3)\\
&\beta_{2365} = \frac{1}{5}(3a_2 - 2a_3) && \beta_{4156} = -a_2\\
&\beta_{1263} = \frac{1}{5}(a_2 + a_3) && \beta_{2463} = -\frac{1}{5}(a_2 + a_3).
\end{aligned}$$

SET 7	*Base*	*Reducing indices*
	Each non-starred term of column 2	All pairs containing 6 but not 8

This gives 40 equations. Precisely the remarks made on Set 6 apply to Set 7, and we obtain ten equations for the $\beta_{(6)}$ in terms of a_2 and a_3. Five of these equations turn out to be identical with five of those from Set 6. The remaining five, however, can be satisfied by (15) only if $a_2=a_3=0$. It follows that $\beta_{(6)}=0$, and from (14) that $\alpha_{(7)}=0$. By a previous remark this implies that $\alpha_{(67)}=\beta_{(67)}=\gamma_{(67)}=0$. We also have that $\alpha_{(8)}=0$.

To sum up, we have shown that every variable is zero, except those of the (68), (78), and (678) sets.

SET 8	*Base*	*Reducing indices*
	Every non-starred term of column 1	All pairs containing 8 but not 7

This set of 40 equations has the same form as Set 6, and shows at once that $\alpha_{(68)}=\beta_{(68)}=\gamma_{(68)}=0$.

SET 9	*Base*	*Reducing indices*
	Every non-starred term of column 2	All pairs containing 8 but not 6

This set of 40 equations has the same form as Set 7, and shows at once that $\alpha_{(78)}=\beta_{(78)}=\gamma_{(78)}=0$.

SET 10	*Base*	*Reducing indices*
	Every non-starred term of column 2 transposed	All pairs containing one 7

This set of 60 equations gives immediately $\alpha_{(678)}=\beta_{(678)}=\gamma_{(678)}=0$.

We have now shown that the reduction equations imply that each of the 210 variables is zero. This is a solution since the 840 reduction equations are homogeneous in the 210 variables. Thus there exists no $\overline{A}_{(8)}$.

PROPOSITION III. *There exists no $\overline{A}_{2m}$ for $m>4$.* (Cf. §40.)

From (107) we see that

$$A_{2m} = \sum g_{(2)} \overbrace{\cdots}^{m-2} g_{(2)}(W_{(4)} + \breve{W}_{(4)}).$$

By Theorem 10 of §31 this expression contains $C_4^{2m}N(m-2)$ constant coefficients. There are C_2^{2m} first reductions of A_{2m}, each of which contains $2C_4^{2m-2}N(m-3)$ coefficients which are linear homogeneous functions with constant coefficients of the coefficients of A_{2m}; the factor 2 is introduced

by the presence of $W_{(4)}$ and $\breve{W}_{(4)}$. If A_{2m} is to be an $\bar{A}_{2m}$ then all its first reductions must be identically zero; if we use induction, assuming that the proposition is correct for $m-1$, then every coefficient of every first reduction of A_{2m} is zero. We are thus led to a set of linear homogeneous equations with constant coefficients in the $C_4^{2m}N(m-2)$ constants of A_{2m}, the number of such equations being $2C_2^{2m}C_4^{2m-2}N(m-3)$. Thus the ratio of the number of equations to the number of unknowns is $2(m-2)$; the conditions which these variables must satisfy become stronger as m increases. Since for $m=4$ these conditions are so strong that all the variables must be zero, it seems probable that with m still greater the same will be true, so that there exists no $\bar{A}_{2m}$ for $m>4$. We have not been able to construct a rigorous proof of this.

APPENDIX V

We shall here prove the theorem mentioned in §42, namely:

THEOREM. *The equation $A_{2m}=0$ is equivalent to the set of equations obtained by equating to zero all the $(m-3)$rd reductions of A_{2m}.*

By definition, (107),

$$(1) \qquad A_{2m} \equiv \sum g_{(2)} \overbrace{\cdots}^{m-2} g_{(2)}(W_{(4)} + \breve{W}_{(4)}).$$

LEMMA I. *For every A_{2m-2} not identically zero, with $m>3$, there exists exactly one A_{2m} such that every first reduction of A_{2m} is identically A_{2m-2}; this A_{2m} is a sum of terms each of which is a product of $g_{(2)}$ and A_{2m-2}.*

Let

$$(2) \qquad \overset{*}{A}_{2m} = a_{12}g_{r_1,r_2}A_{r_3,\ldots,r_{2m}} + \cdots + a_{2m-1,2m}g_{r_{2m-1},2m}A_{r_1,\ \ ,r_{2m-2}},$$

where $a_{12}, \cdots, a_{2m-1,2m}$ are unknown constants; this is clearly a tensor of the form (1). Let us now equate to A_{2m-2} the first reductions of (2); we thus have the equations

$$(3) \qquad \sum g^{(2)}\overset{*}{A}_{2m} = A_{2m-2}\ ;$$

these are C_2^{2m} linear equations in the C_2^{2m} unknowns a_{ij}. Let us now consider the set of equations

$$(4) \qquad \sum g^{(2)}\overset{*}{A}_{2m} = 0,$$

with the same left members as (3). If the determinant of (4), considered as linear equations in the a_{ij}, were zero then (4) would have a solution with the a_{ij} not all zero; then because of the difference in indices in the various terms of (2) there would be a tensor A_{2m} not identically zero but having

all its first reductions identically zero, that is an $\overline{A}_{2m}$. But by §40 there exists no $\overline{A}_{2m}$ for $m>3$ if we accept Proposition III. Hence the determinant of (4) is not zero. Since the determinants of (3) and (4) are the same the determinant of (3) is not zero. Hence there exists a unique solution (2) of the equations (3).

Now suppose A_{2m} is a tensor of form (1) such that every first reduction of A_{2m} is identically A_{2m-2}; since A^*_{2m} has the same property it follows that every first reduction of the tensor $A_{2m}-A^*_{2m}$ is identically zero; hence by the Theorem and the Proposition of §40 it follows that A_{2m} is identically A^*_{2m}. We thus have proved the lemma.

In a similar way we can prove

LEMMA II. *For every set of $k \leqq C_2^{2m}$ tensors $\overset{(k)}{A}_{2m-2}$ not all identically zero, with $m>3$, there exists exactly one A_{2m} such that the first reductions of A_{2m} are exactly the set $\overset{(k)}{A}_{2m-2}$ with specified indices; this A_{2m} is a sum of terms each of which is a product of $g_{(2)}$ and one of $\overset{(k)}{A}_{2m-2}$.*

We are now ready to prove the Theorem. Consider the equation

$$A_{2m} = 0. \tag{5}$$

Let the first reductions of A_{2m} be

$$\overset{(k)}{A}_{2m-2} = \sum g^{(2)} A_{2m}, \qquad (k = 1, \cdots, C_2^{2m}). \tag{6}$$

Then by Lemma II,

$$A_{2m} = \sum_k g_{(2)} \overset{(k)}{A}_{2m-2}. \tag{7}$$

Consider the set of equations

$$\overset{(k)}{A}_{2m-2} = 0, \qquad (k = 1, \cdots, C_2^{2m}). \tag{8}$$

By (6) it follows that (5) implies (8); by (7) it follows that (8) implies (5); hence (5) and (8) are equivalent. Now the theorem follows readily by induction. It is to be noted that to prove this theorem we have used Proposition III (p. 123).

APPENDIX VI

We consider here, as stated in the footnote to page 88, the application to the solar field of the tensor law

$$S_{(6)} = 0. \tag{155}$$

We saw that the equation (134), $W_{(4)}=0$, represents a necessary and sufficient condition that ds^2 be reducible to the form

$$ds^2 = h\left[-\sum_{i=1}^{3} dx_i^2 + dx_4^2\right], \tag{160}$$

where h is a function of the x_i. It seems probable that there exists a similar theorem stating that (155) is a necessary and sufficient condition that ds^2 be reducible to the form

$$ds^2 = -h_1 \sum_{i=1}^{3} dx_i^2 + h_2 dx_4^2, \tag{212}$$

where h_1 and h_2 are functions of the x_i. We have not had the opportunity to attempt a proof of this proposition, but we have proved the following: *Every* ds^2 *reducible to the form* (162) *is a solution of* (155).

This theorem is adequate for a study of the equation $S_{(6)}=0$ as a possible law of gravitation applied to the solar field. It tells us that every set $g_{\mu\nu}$ of the form required by a static, radially symmetric field is a solution. Thus, we have an even greater arbitrariness than in the case of $W_{(4)}=0$. In particular the constants a_1, b_1, and b_2 of (165), which control the period of revolution, the motion of the perihelion point of a planet and the deflection of a light ray in passing a heavy mass, are arbitrary.

INDEX